A PRIORI C^2 ESTIMATES FOR FULLY NONLINEAR ELLIPTIC EQUATIONS ON RIEMANNIAN MANIFOLDS

The aim of the talk is to discuss the main results in the preprint [G] by Bo Guan. He considered a class of fully nonlinear equation on a compact (with or without boundary). In the case of a domain in $\mathbb{R}^n$, this type of equations was consider by Caffarelli, Nirenberg and Spruck in their seminal paper [CNS].

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 2$. The boundary ∂M is smooth. Let f be a smooth symmetric function of n variables and χ a smooth $(0, 2)$ tensor on $\bar{M}$. The fully nonlinear equations under consideration is of the form

$$f(\lambda([\nabla^2 u + \chi])) = \psi \text{ in } M, \tag{0.1}$$

where $\nabla^2 u$ denotes the Hessian of $u \in C^2(M)$ and $\lambda(\nabla^2 + \chi) = (\lambda_1, ..., \lambda_n)$ are the eigenvalues of $\nabla^2 u + \chi$ w.r.t the metric g.

Assume that f is defined in a symmetric open and convex cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin and $\partial\Gamma \neq \emptyset$,

$$\Gamma^+ \equiv \{\lambda \in \mathbb{R}^n : \lambda_i > 0 \ \ \forall i = 1, ..., n\} \subseteq \Gamma, \tag{0.2}$$

and f satisfies the standard structure conditions

$$f_{\lambda_i} = \frac{\partial f}{\partial \lambda} > 0 \text{ in } \Gamma, \quad 1 \leq i \leq n, \tag{0.3}$$

$$f \text{ is concave function.} \tag{0.4}$$

Moreover, for $\sigma > 0$, one defines $\Gamma^\sigma = \{\lambda \in \Gamma : f(\lambda) > \sigma\}$. Suppose that Γ^σ is non empty. Let $\mathcal{C}_\sigma$ denote the tangent cone at infinity to the level surface $\partial\Gamma^\sigma$. Let $\mathcal{C}_\sigma$ be the open component of $\Gamma \setminus (\mathcal{C}_\sigma \cap \Gamma)$ containing Γ^σ.

The main theorem which we are discussing is

Theorem 0.1. *Let* $\psi \in C^2(M \times \mathbb{R}) \cap C^1(\bar{M} \times \mathbb{R})$ *and* $u \in C^4(M) \cap C^2(\bar{M})$ *be an admissible solution of* (0.1)*. Suppose* $a \leq u \leq b$ *on* $\bar{M}$ *and let*

$$\underline{\psi}(x) = \min_{a \leq z \leq b} \psi(x, z), \ \ \hat{\psi}(x) = \max_{a \leq z \leq b} \psi(x, z), \ \ x \in \bar{M}.$$

In addition to (0.3)-(0.4)*, assume*

$$\delta_{\underline{\psi}, f} = \inf_{\bar{M}} \underline{\psi} - \sup_{\partial\Gamma} f > 0. \tag{0.5}$$

and that there exists a function $\underline{u} \in C^2(\bar{M})$ *satisfying*

$$\lambda[\nabla^2 \underline{u} + \chi](x) \in \mathcal{C}^+_{\hat{\psi}(x)}, \quad \forall\, x \in \bar{M}. \tag{0.6}$$

Then

$$\max_{\bar{M}} |\nabla^2 u| \leq C_1 \big(1 + \max_{\partial M} |\nabla^2 u|\big).$$

In particular, if M *is closed* ($\partial M = \emptyset$) *then*

$$|\nabla^2 u| \leq C_2 e^{C_3(u - \inf_M u)} \quad \text{on } M$$

where C_1, C_2 *depend on* $|u|_{C^1(M)}$ *but not on* $1/\delta_{\underline{\psi},f}$ *and* C_3 *is a uniform constant (independent of* u*).*

References

[CNS] L. Caffarelli, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian, *Acta Math.* **155** (1985), 261-301.

[G] B. Guan, Second order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds, *preprint* arXiv1211.0181v1