Some properties of squeezing functions on bounded domains

(based on the paper by F. Deng, Q. Guan, L. Zhang)

Definition 1. Let $D \subset \mathbb{C}^N$ be a bounded domain. The squeezing function of D is the function $s_D : D \to (0, 1]$ given by

$$s_D(x) := \sup\{r > 0 : \exists\, f \in \mathcal{O}(D, \mathbb{B}_N) : f(x) = 0,\ \mathbb{B}_N(0, r) \subset f(D),\ f \text{ is injective}\},\ x \in D,$$

where $\mathbb{B}_N(a, r)$ denotes the euclidean ball with radius r and center a, and $\mathbb{B}_N = \mathbb{B}_N(0, 1)$.

Lemma 2. *Let $D \subset \mathbb{C}^N$ be a bounded domain, $a \in \mathbb{C}^N$, $x \in D$, and let $(f_j)_{j\in\mathbb{N}} \subset \mathcal{O}(D, \mathbb{C}^N)$ be a sequence of injective mappings with $f_j(x) = a$. If f_j are compactly uniformly convergent on D to some holomorphic map $f : D \to \mathbb{C}^N$, and $\operatorname{int} \bigcap_{j\in\mathbb{N}} f_j(D) \neq \varnothing$, then f is injective.*

Theorem 3. *If $D \subset \mathbb{C}^N$ is a bounded domain, then for every $x \in D$ there exists an extremal map for s_D at x, i.e. an injective holomorphic map $f : D \to \mathbb{B}_N$ s.t. $f(x) = 0$ and $\mathbb{B}_N(0, s_D(x)) \subset f(D)$.*

Observation 4 (Properties of squeezing functions)**.** *Let $D, G \subset \mathbb{C}^N$ be bounded domains. Then*

- *If $\varphi : D \to G$ is biholomorphic, then $s_G \circ \varphi = s_D$.*
- *If $Aut(D)$ acts transitively, then s_D is constant.*
- $s_{\mathbb{B}_N} \equiv 1$.
- *s_D is continuous.*
- *If $s_D(y) = 1$ for some $y \in D$, then D is biholomorphcally equivalent to $\mathbb{B}_N$.*
- *If $A \subsetneq D$ is an analytic subset, then*

$$s_{D\setminus A}(x) \le \tanh\left(\frac{1}{2}\operatorname{dist}_{c_D}(x, A)\right),$$

  where $\operatorname{dist}_{c_D}$ is the distance from x to A w.r.t. the Carathéodory distance c_D in D.

Proposition 5. *Let $D \subset \mathbb{C}^N$ be a bounded domain. If $\inf_{x\in D} s_D(x) > 0$, then (D, c_D) is a complete metric space. In particular, D is pseudoconvex.*

Proposition 6. *Let $D \subset \mathbb{C}^N$ be a bounded domain, $x \in D$, $Y \in \mathbb{C}^N$. Then*

$$s_D(x)\, \gamma_D(x, Y) \le \kappa_D(x, Y) \le \frac{1}{s_D(x)}\gamma_D(x, Y),$$

where γ_D and κ_D are the Carathéodory-Reiffen and Kobayashi-Royden metrics in D, respectively.

Proposition 7. *Let $D \subset \mathbb{C}$ be a bounded domain. Then*

$$\gamma_D(x, 1) \ge \frac{s_D(x)}{4\operatorname{dist}(x, \partial D)},\ x \in D.$$

Proposition 8. *Let $D \subset \mathbb{C}$ be a bounded finitely connected domain, and let E be a connected component of $\widehat{\mathbb{C}} \setminus E$. Then:*

1. *If E is not a single point, then*
$$\lim_{D\ni x\to E} s_D(x) = 1.$$
2. *If E is a single point, then*
$$\lim_{D\ni x\to E} s_D(x) = 0.$$