$\mathbb{R}$–linear problem and its applications

Vladimir V. Mityushev

Dept. Comp. Sci. & Comp. Meth., Pedagogical University, Krakow, Poland

Sisyphe, UPMC 6, Paris, France

Various **boundary value problems** are reduced to singular integral equations [5, 13, 14]. Only some of them can be solved in closed form. In the present talk, we follow the lines of [7, 8, 10] and describe solution to the $\mathbb{R}$–linear problem for multiply connected domains on the complex plane $\mathbb{C}$ which in a particular case yields the Riemann–Hilbert problem.

Let D be a multiply connected domain whose boundary ∂D consists of simple closed Lyapunov curves. The positive orientation on ∂D leaves D to the left. The **scalar linear Riemann–Hilbert problem** for D is stated as follows. Given Hölder continuous functions $\lambda(t) \neq 0$ and $f(t)$ on ∂D. To find a function $\phi(z)$ analytic in D, continuous in the closure of D with the boundary condition

$$Re\ \overline{\lambda(t)}\phi(t) = f(t),\ t \in \partial D. \tag{1}$$

Let $D_k \cup \partial D_k$ (holes) complement D to the extended complex planes. Given Hölder continuous functions $a(t) \neq 0$, $b(t)$ and $f(t)$ on ∂D. To find a function $\phi(z)$ analytic in $\cup_{k=1}^{n} D_k \cup D$, continuous in $D_k \cup \partial D_k$ and in $D \cup \partial D$ with the $\mathbb{R}$–**linear condition**

$$\phi^+(t) = a(t)\phi^-(t) + b(t)\overline{\phi^-(t)} + f(t),\ t \in \partial D. \tag{2}$$

Here $\phi^+(t)$ is the limit value of $\phi(z)$ when $z \in D$ tends to $t \in \partial D$, $\phi^-(t)$ is the limit value of $\phi(z)$ when $z \in D_k$ tends to $t \in \partial D$. In the case $|a(t)| \equiv |b(t)|$ the $\mathbb{R}$–linear problem is reduced to the Riemann–Hilbert problem (1).

These problems can be considered as a generalization of the classical **Dirichlet and Neumann problems** for harmonic functions. Mixed boundary value problems constitute a partial case of (1). One knows the famous Poisson formula which solves the Dirichlet problem for a disk. The exact solution of the Dirichlet problem for a circular annulus is also known due to Villat–Dini. There are exact formulae for circular multiply connected domains with geometrical restrictions (see [2, 3] and papers cited therein). Formulae from [7, 8, 10] can be considered as a generalization of the Poisson, Villat–Dini and Crowdy - DeLillo - Driscoll - Elcrat - Pfaltzgraff [2, 3] formulae to arbitrary circular multiply connected domains.

In order to deduce our formulae we first reduce the boundary value problem to the $\mathbb{R}$–linear problem and solve the later one by use of functional equations. By *functional equations* we mean iterative functional equations with shift into domain. Hence, we do not use traditional integral equations and infinite systems of linear algebraic equations. The solution is given explicitly in terms of the known functions or constants and geometric parameters of domains. In particular, the almost uniform convergence of the Poincaré θ_2–series for any multiply connected domain is proved [7]. Applications of the $\mathbb{R}$-problem to the explicit Schwarz-Christoffel formula [10] in general case, to composites [4, 6, 9, 11, 12] and to RVE [4] are discussed.

References

[1] B. Bojarski, Soobsch. AN GruzSSR **25** , n. 4 (1960) 385.

[2] D. Crowdy, Nonlinearity **21**, (2008) T205-T219.

[3] T.K. DeLillo, T.A. Driscoll, A.R. Elcrat, J.A. Pfaltzgraff, Comput. Methods Funct. Theory **6** (2006) 301.

[4] R. Czapla, W. Nawalaniec, V. Mityushev, Comput. Mat. Sci. 63 (2012) 118.

[5] F.D. Gakhov: *Boundary Value Problems.* Nauka, Moscow, 1977.

[6] P. Jarczyk, V. Mityushev, Proc. Roy. Soc. London 468A (2012) 954.

[7] V.V. Mityushev, Proceedings Amer.Math.Soc. **126** (1998), n. 8, 2399.

[8] V.V. Mityushev, S.V. Rogosin: *Constructive methods to linear and non-linear boundary value problems of the analytic function. Theory and applications*, Chapman & Hall / CRC, Boca Raton etc. 2000.

[9] V. Mityushev, N. Rylko, Multiscale Model. Simul. 10 (2012) 180.

[10] Mityushev V., Comput. Methods Funct. Theory; 11, (2011); 12, (2012), 449.

[11] V. Mityushev, N. Rylko, Math. Comp. Modelling 57, (2013) 1350.

[12] V. Mityushev, N. Rylko Quart. J. Mech. Appl. Math. (2013)

[13] N.I. Muskhelishvili: *Singular Integral Equations.* Nauka, Moscow, 1968.

[14] I.N. Vekua: *Generalized Analytic Functions.* Nauka, Moscow, 1988.