Complex geodesics in strictly linear convex domains

THEOREM 0.1. Let $D \Subset \mathbb{C}^n$ be a strictly linear convex domain with the boundary of class $\mathcal{C}^{k+1,s}$, where $k \in \mathbb{N}$, $s \in (0,1)$. Then the mapping

$$D \times \mathbb{C}^m \setminus \{0\} \ni (w,v) \mapsto f_{w,v} \in Y_m,$$

where $f_{w,v}$ is a complex geodesic through the point w and in direction v, is of class $\mathcal{C}^{k-1,s}$. (By Y_m we denote the subspace of those mappings in $\mathcal{C}^{0,s}(\mathbb{T},\mathbb{C}^m)$ that admit a holomorphic continuation to $\mathbb{D}$.)

The proof is to apply the implicit function theorem to the function

$$\Xi : \mathbb{C}^m \times \mathbb{C}^m \times Y_m \times \mathbb{R} \longrightarrow T \times Y'_{m-1} \times \mathbb{C}^m \times \mathbb{C}^m$$

given by

$$\Xi(w,v,f,\lambda) := \left(r \circ f, \pi\left(\frac{[H^t \tau \circ f]}{(H^t \tau \circ f)_1}\right), f(0)-w, f'(0)-\lambda v\right),$$

where τ is a vector field that agrees with the derivative of the defining function r_z on ∂D, and H is a matrix whose entries are holomorphic functions on the disc, defined to simplify $F' \bullet \tilde{F}$. F is a fixed extremal mapping for $W \in D$ and $V \in \mathbb{C}^n$ and $\tilde{F}$ is an associated mapping.

The most important is the following

LEMMA 0.2. Let $W : \mathbb{T} \to \mathcal{M}_m^{-1}$ be a map of class $\mathcal{C}^{0,s}$ such that $W(\zeta)$ is self-adjoint and positive definite matrix for every $\zeta \in \mathbb{T}$. Then there exists $H : \overline{\mathbb{D}} \to \mathcal{M}_m^{-1}$ satisfying $H \in \mathcal{C}^{0,s}(\overline{\mathbb{D}}) \cap \mathcal{O}(\mathbb{D})$ and $H^*H = W$ on $\mathbb{T}$.

The proof of Lemma consists of four steps. First, we show it for $w := W - I$ with sufficiently small $\mathcal{C}^{0,s}$ norm. Next, using this fact, we prove that for arbitrary $\delta > 0$, there exists $m \times m$ matrix valued function $H \in \mathcal{C}^{0,s-\delta}(\overline{\mathbb{D}}) \cap \mathcal{O}(\mathbb{D})$ such that $H^*H = W$ on $\mathbb{T}$. The third step is, that for $W : \mathbb{T} \to \mathcal{M}_m^{-1}$ of class $\mathcal{C}^{0,s}$ and arbitrary point $\omega \in \mathbb{T}$, there exists $\widetilde{W} : \mathbb{T} \to \mathcal{M}_m^{-1}$ such that $\widetilde{W} = W$ near ω and the equation $H^*H = \widetilde{W}$ admits a solution $H \in \mathcal{C}^{0,s}(\overline{\mathbb{D}}) \cap \mathcal{O}(\mathbb{D})$. Finally, we prove that $H \in \mathcal{C}^{0,s-\delta}(\overline{\mathbb{D}}) \cap \mathcal{O}(\mathbb{D})$ satysfying $H^*H = W$ on $\mathbb{T}$ is actually in $\mathcal{C}^{0,s}(\overline{\mathbb{D}})$.