DELTA PLURISUBHARMONIC FUNCTIONS

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Let $\Omega \subseteq \mathbb{C}^n$, $n \geq 1$, be a bounded hyperconvex domain. We say that a plurisubharmonic function φ defined on Ω belongs to $\mathcal{E}_0$ if $\lim_{z\to\xi}\varphi(z) = 0$, for every $\xi \in \partial\Omega$, and $\int_\Omega (dd^c\varphi)^n < +\infty$. Let us define for $p > 0$

$$\mathcal{E}_p = \{u \in \mathcal{PSH}(\Omega) : \exists u_j \in \mathcal{E}_0; u_j \searrow u, \sup_j e_p(u_j) < \infty\},$$

where $e_p(u) = \int_\Omega (-u)^p (dd^c u)^n$. Along with $\mathcal{E}_p$ we are also interested in the following set of measures

$$\mathcal{M}_p = \big\{\mu \ : \mu \text{ is a positive Radon measure on } \Omega \text{ such that } (dd^c u)^n = \mu \text{ for some } u \in \mathcal{E}_p\big\}\,.$$

Theorem 0.1. (1) *The ordered vector space $(\delta\mathcal{E}_p, \geq)$ is a Riesz space with the supremum, and infimum defined as*

$$\sup(u,v) = \sup(u_1 - u_2, v_1 - v_2) = \max(u_1 + v_2, u_2 + v_1) - (u_2 + v_2) \text{ and}$$
$$\inf(u,v) = -\sup(-u,-v)\,,$$

(2) *The ordered vector space $(\delta\mathcal{E}_p, \succcurlyeq)$ is not Riesz space for $n > 1$, where $\succcurlyeq$ is the order generated by the cone $\mathcal{E}_p$ (i.e. $u \succcurlyeq v$ if and only if $u - v \in \mathcal{E}_p$).*

(3) *The ordered vector spece $(\delta\mathcal{M}_p, \succcurlyeq)$ is a Riesz space, where $\succcurlyeq$ is the order generated by the cone $\mathcal{M}_p$ and supremum and infimum of two positive measures μ, ν are defined by*

$$\sup(\mu,\nu)(A) = \sup\{\mu(B) + \nu(A \setminus B) : B \subset A\},$$
$$\inf(\mu,\nu)(A) = \inf\{\mu(B) + \nu(A \setminus B) : B \subset A\}.$$

Let $p > 0$. Then for $u \in \delta\mathcal{E}_p$ we define:

$$\|u\|_p = \inf_{\substack{u_1 - u_2 = u\\ u_1, u_2 \in \mathcal{E}_p}} \left(\int_\Omega (-(u_1 + u_2))^p (dd^c(u_1 + u_2))^n\right)^{\frac{1}{n+p}}\,,$$

and let $|\cdot|_p$ be defined on $\delta\mathcal{M}_p$ by

$$|\mu|_p = \inf_{\substack{\mu_1 - \mu_2 = \mu\\ \mu_1, \mu_2 \in \mathcal{M}_p}} \|u_{\mu_1}\|_p^n + \|u_{\mu_2}\|_p^n\,,$$

where $u_{\mu_j} \in \mathcal{E}_p$, $j = 1, 2$, are the uniquely determined solutions to the equations $\left(dd^c u_{\mu_j}\right)^n = \mu_j$.

Theorem 0.2. *Let $p > 0$.*

a) $(\delta\mathcal{E}_p, \|\cdot\|_p)$ *is a quasi-Banach space for $p \neq 1$, and $(\delta\mathcal{E}_1, \|\cdot\|_1)$ is a Banach spaces.*

b) $(\delta\mathcal{M}_p, |\cdot|_p)$ *is a quasi-Banach space for $p \neq 1$, and $(\delta\mathcal{M}_1, |\cdot|_1)$ is a Banach space.*

The set of all regular linear functional defined on $(X, \succcurlyeq)$, is denoted by X^r, and the set of ordered bounded linear functional is denoted by X^b. These vector subspaces satisfy $X^r \subseteq X^b \subseteq X^*$.

The *topological dual* of a topological vector space (X, τ) is denoted by X', i.e. X' is the vector subspace of X^* consisting of all τ-continuous functionals. Let $\mathcal{K}$ be any cone in (X, τ), then we define the *dual cone* $\mathcal{K}'$ of $\mathcal{K}$ in (X, τ) by

$$\mathcal{K}' = \{f \in X^* : f(x) \geq 0, \text{ for each } x \in \mathcal{K}\}\,.$$

Let Ω be bounded hyperconvex domain in $\mathbb{C}^n$. For each non-pluripolar set $\omega \Subset \Omega$ we define

$$D_\omega : \mathcal{E}_p \ni u \to \int_\omega \Delta u \in \mathbb{R}\,.$$

Then D_ω is a positive, and linear, functional defined on $\mathcal{E}_p$. Hence, it can be extended to a regular, linear functional, defined on $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)$. Let $\mathcal{D}$ denote the family the functionals D_ω together with the zero functional, i.e.

$$\mathcal{D} = \{0\} \cup \{D_\omega : \omega \Subset \Omega\}\,.$$

Theorem 0.3. *Let $p > 0$. Then*

(1) *$\mathcal{E}_p$ is a normal cone in $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)$,*

(2) *$(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)^r = (\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)^b = (\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)'$, i.e.*
$$(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)' = \mathcal{E}_p' - \mathcal{E}_p'\,.$$

(3) *The vector space $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)'$, $p \geq 1$, is equal to the closure of $\delta\mathcal{M}_p$ in $\sigma((\delta\mathcal{E}_p)', \delta\mathcal{E}_p)$.*

(4) *The vector space $(\delta\mathcal{E}_p, \succcurlyeq, \|\cdot\|_p)'$ is equal to the $\sigma((\delta\mathcal{E}_p)', \delta\mathcal{E}_p)$-closure of the linear span of $\mathcal{D}$, where*

The correspondent statements for parts (1)-(3) are also true for $(\delta\mathcal{M}_p, \succcurlyeq, \|\cdot\|_p)$.

We are interested in extension of the complex Monge-Ampère operator to the class of delta plurisubharmonic functions. One direction could be to follow Cegrell and Wiklund idea and define for $u \in \delta\mathcal{E}$

$$\mathrm{MA}_1(u) = \mathrm{MA}_1(u_1 - u_2) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (dd^c u_1)^k \wedge (dd^c u_2)^{n-k}\,.$$

One the other hand we can also define for $u = u_1 - u_2 \in \delta\mathcal{E}$ we can define

$$u^- = \left(\sup\{\beta \in \mathcal{E} : \text{ there exists } \alpha \in \mathcal{E} \text{ such that } u_1 + \beta = u_2 + \alpha\}\right)^*,$$

and

$$u^+ = \left(\sup\{\alpha \in \mathcal{E} : \text{ there exists } \beta \in \mathcal{E} \text{ such that } u_1 + \beta = u_2 + \alpha\}\right)^*.$$

It can be proved that $u = u^+ - u^-$.

Now we can extend the complex Monge-Ampère by the formula

$$\mathrm{MA}_2(u) = \mathrm{MA}_2(u^+ - u^-) = (dd^c u^+)^n - (dd^c u^-)^n\,.$$

The Dirichlet problem (existence and uniqueness of a solution) for operators MA_1 and MA_2 is an open problem. Some partial result are obtained in the case of radially symmetric plurisubharmonic functions.