ON SOME PROPERTIES OF THE SZEGÖ AND POISSON-SZEGÖ KERNELS

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In the following, Ω denotes the bounded domain in R^N or C^n. Moreover, when we consider the Poisson kernel and Szegö kernel, we also assume, that Ω has C^2 boundary.

Let $F\colon \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ be a fundamental solution for the Laplacian in $\mathbb{R}^N$ i.e. let $\Delta F = \delta_0$, where δ_0 denotes the Dirac distribution; in other words the equality $\int_{\mathbb{R}^N}(\Delta\phi)F\,dV = \phi(0)$ holds for every test function $\phi \in C_c^\infty(\mathbb{R}^\mathbb{N})$. The explicit form of F is well known, namely $F = (2\pi)^{-1}\log||x||$ for $N = 2$ and $F = (2-N)^{-1}\omega_{N-1}^{-1}||x||^{2-N}$ for $N \neq 2$, where ω_{N-1} denotes the area of the unit sphere in $\mathbb{R}^N$. Now let us define $\Gamma\colon \mathbb{R}^N\times\mathbb{R}^N\setminus\{\text{diag}\} \to \mathbb{R}$ as $\Gamma(x,y) := F(x-y)$.

Def. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2 boundary. If a function $G\colon \Omega\times\overline{\Omega}\setminus\{\text{diag}\} \to \mathbb{R}$ satisfies the following conditions:

- G is of class C^2 on $\Omega\times\Omega\setminus\{\text{diag}\}$
- G is of class $C^{2-\epsilon}$ on $\Omega\times\overline{\Omega}\setminus\{\text{diag}\}$ for every small, positive ϵ
- $G(x,\cdot)$ is harmonic on $\Omega\setminus\{x\}$ for every $x \in \Omega$
- $G(x,\cdot) - \Gamma(x,\cdot)$ extends to a harmonic function on Ω for every $x \in \Omega$
- $G(x,\cdot)$ vanishes on the boundary of Ω

then G is called a **Green's function** of the domain Ω.

Thm. [6, ch. 1] *G is symmetric on $\Omega\times\Omega$, uniquely determined by the above conditions and exists in the case of a C^2 class of a boundary of Ω.*

Let n denotes the unit, normal, outward-pointing vector field on the boundary of Ω and define $\mathbf{P}\colon \Omega\times\partial\Omega \to \mathbb{R}$, the **Poisson kernel** of Ω [6, ch. 1], as follows:

$$\mathbf{P}(x,y) := \nabla_{-n_y}G(x,\cdot) \quad \text{for}\,(x,y) \in \Omega\times\partial\Omega,$$

where n_y denotes the value of n at y.

Thm. [4, ch. 1] *$\mathbf{P}$ is strictly positive, $\mathbf{P}(\cdot,y)$ is harmonic and the following Poisson integral formula holds for every complex function u, harmonic on Ω and continuous on $\overline{\Omega}$:*

$$u(x) = \int_{\partial\Omega}\mathbf{P}(x,y)u(y)d\sigma(y),$$

where $d\sigma$ is a standard area measure [4, App. II] *on the boundary $\partial\Omega$.*

Now let Ω be a bounded domain in $\mathbb{R}^N$, with C^2 boundary and let $\phi\colon \mathbb{R} \to [0,1]$ be a cut-of function, supported in $[-2,2]$, with $\phi \equiv 1$ on $-1,1$. Moreover, let $\delta_\Omega(x) \equiv \text{dist}(x,\partial\Omega)$. It is possible to show [4, ch. 8], that for every sufficiently small, positive ϵ_0, the function

$$\lambda(x) = \begin{cases} -[\phi(|x|/\epsilon_0)\delta_\Omega(|x|) + (1-\phi(|x|/\epsilon_0))] & \text{for } x \in \overline{\Omega}, \\ \phi(|x|/\epsilon_0)\delta_\Omega(|x|) + (1-\phi(|x|/\epsilon_0)) & \text{for } x \in \mathbb{R}^N\setminus\Omega \end{cases}$$

is a C^2 defining function [3, ch. 1] for Ω. Moreover, if $0 < \epsilon < \epsilon_0$, then $\partial\Omega_\epsilon \equiv \{x \in \Omega : \lambda(x) = -\epsilon\}$ is a C^2 regular submanifold in $\mathbb{R}^N$ that bounds $\Omega_\epsilon \equiv \{x \in \Omega : \lambda(x) < -\epsilon\}$. In the following, we shall write $d\sigma$ instead of $d\sigma_\epsilon$ (the area measure on $\partial\Omega_\epsilon$.) Now let us define the space $h^2(\Omega)$ in the following way:

$$h^2(\Omega) := \{f \text{ complex-valued, harmonic on } \Omega : \sup_{0<\epsilon<\epsilon_0} \Big[\int_{\partial\Omega_\epsilon} |f(x)|^2 d\sigma(x) \Big]^{\frac{1}{2}} < \infty\}$$

This definition is in fact independent of the particular choice of the defining function, for which the boundaries of the approximating domains Ω_ϵ are C^2 smooth (Lemma of E. Stein, [4, ch. 8])

Thm. [4, ch. 8] *Let Ω be a bounded domain in $\mathbb{R}^N$, with C^2 boundary and let f be a complex-valued, harmonic function on Ω. The following conditions are equivalent*:
1. $f \in h^2(\Omega)$
2. *there exists a complex-valued function $\widetilde{f} \in L^2(\partial\Omega, d\sigma)$, such that*

$$f(x) = \int_{\partial\Omega} \mathbf{P}(x,y)\widetilde{f}(y) d\sigma(y) \quad \textit{for } x \in \Omega$$

3. $|f|^2$ *has a harmonic majorant.*
Moreover, if the above conditions are satisfied, then $\widetilde{f}$ is equal almost everywhere on the boundary $\partial\Omega$ to the boundary values of f i.e.

$$\lim_{\epsilon \to 0^+} f(y - \epsilon n_y) = \widetilde{f}(y) \quad \textit{for almost all } y \in \partial\Omega$$

In the following we shall denote the boundary values by a tilde.

Now, assume that Ω is a bounded domain in C^n and define the Bergman spaces and Hardy spaces (in the following, for the purpose of integration, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$):

$$A^2(\Omega) := \{f \in \mathcal{O}(\Omega) : \Big[\int_\Omega |f(z)|^2 dV(z) \Big]^{\frac{1}{2}} = ||f||_{A^2(\Omega)} < \infty\}$$

$$H^2(\Omega) := \{f \in \mathcal{O}(\Omega) : \sup_{\epsilon>0} \Big[\int_{\partial\Omega_\epsilon} |f(z)|^2 d\sigma(z) \Big]^{\frac{1}{2}} = ||f||_{H^2(\Omega)} < \infty\}$$

Thm. [4, ch. 8] *Let $f \in H^2(\Omega)$ and let $\widetilde{f}$ denotes its boundary values. Then*

$$||f||_{H^2(\Omega)} = \sup_{\epsilon>0} \Big[\int_{\partial\Omega_\epsilon} |f(z)|^2 d\sigma(z) \Big]^{\frac{1}{2}} = \Big[\int_{\partial\Omega} |\widetilde{f}(z)|^2 d\sigma(z) \Big]^{\frac{1}{2}} = ||\widetilde{f}||_{L^2(\partial\Omega)}$$

Now let us define the scalar product in $A^2(\Omega)$ and $H^2(\Omega)$

$$< f, g >_{A^2(\Omega)} := \int_\Omega f(z)\overline{g(z)} dV(z), \quad < f, g >_{H^2(\Omega)} := \int_{\partial\Omega} \widetilde{f}(z)\overline{\widetilde{g}(z)} d\sigma(z)$$

Thm. [4,5] *$A^2(\Omega)$ and $H^2(\Omega)$ are separable Hilbert spaces.*

Now let $z \in \Omega$ and let us consider the evaluation functional $f \to E_z(f) := f(z)$ on each of these spaces.

Thm. [4,5] *For every $z \in \Omega$ E_z is continuous, linear functional on each of the spaces $A^2(\Omega), H^2(\Omega)$*

The Riesz representation theorem implies, that for every $z \in \Omega$ there exists exactly one $b_z \in A^2(\Omega)$, such that for all $f \in A^2(\Omega)$ we have

$$f(z) = E_z(f) =< f, b_z >= \int_\Omega f(w)\overline{b_z(w)}dV(w) = \int_\Omega \mathbf{B}(z,w)f(w)dV(w),$$

where $\mathbf{B}\colon \Omega \times \Omega \to \mathbb{C}$ is defined by $\mathbf{B}(z,w) = \overline{b_z(w)}$.

Similarly for every $z \in \Omega$ there exists exactly one $s_z \in H^2(\Omega)$, such that for all $f \in H^2(\Omega)$ we have

$$f(z) = E_z(f) =< f, s_z >= \int_{\partial\Omega} \widetilde{f}(w)\overline{\widetilde{s}_z(w)}d\sigma(w) = \int_{\partial\Omega} \widetilde{\mathbf{S}}(z,w)\widetilde{f}(w)d\sigma(w),$$

where $\widetilde{\mathbf{S}}\colon \Omega \times \partial\Omega \to \mathbb{C}$ is defined by $\widetilde{\mathbf{S}}(z,w) = \overline{\widetilde{s}_z(w)}$.

Moreover, let us define the function $\mathbf{S}\colon \Omega \times \Omega \to \mathbb{C}$, as $\mathbf{S}(z,w) = \overline{s_z(w)}$. The function $\mathbf{B}$ is called the **Bergman kernel** for Ω and the function $\mathbf{S}$ is called the **Szegö kernel** for Ω.

Thm. [4,5] *Let $(\phi_j)_{j=1}^\infty$ be an orthonormal basis in $A^2(\Omega)$ (respectively $H^2(\Omega)$). Then we have the equality*

$$\sum_{j=1}^\infty \phi_j(z)\overline{\phi_j(w)} = \mathbf{B}(z,w) \quad (\textit{resp.} = \mathbf{S}(z,w)),$$

where the series on the LHS converges compactly on $\Omega \times \Omega$.

Thm. [4,5] *The Bergman kernel and Szegö kernel are strictly positive on the diagonal and they are conjugate symmetric. Moreover, the Bergman kernel is an element of $A^2(\Omega)$, with respect to the first variable and the analogous result holds for the Szegö kernel.*

The Bergman kernel and Szegö kernel can almost never be explicitly computed. However, they can be calculated asymptotically in a number of instances [5]. The explicit formulas for these kernels [4,5], in the case of the unit ball in C^n, are as follows:

$$\mathbf{B}(z,w) = \frac{n!}{\pi^n}\frac{1}{(1- < z,w >)^{n+1}}, \quad \mathbf{S}(z,w) = \frac{(n-1)!}{2\pi^n}\frac{1}{(1- < z,w >)^n},$$

where $<,>$ denotes the standard scalar product on C^n.

Thm. (Ramadanov) [5] *Let* $(\Omega_j)_{j=1}^{\infty}$ *be an increasing sequence of bounded domains in* $\mathbb{C}^n$ *and let* $\Omega = \cup_{j=1}^{\infty}\Omega_j$ *be also bounded. Then*

$$\mathbf{B}_\Omega(z,w) = \lim_{j\to\infty} \mathbf{B}_{\Omega_j}(z,w),$$

and the convergence is even uniform on compact subsets of $\Omega\times\Omega$.

Def. [3, App. A.1] Fix an integer $k \geq 1$. Consider the collection $\mathcal{D}_k$ of all bounded domains in $\mathbb{R}^N$, with C^k boundary. Fix one such domain Ω_0. Associate to it the embedding ι_0 into $\mathbb{R}^N$, given by $\iota_0(x) = x$. In other words, ι_0 is a canonical inclusion. Now let $\epsilon > 0$. Set

$$\mathcal{U}_\epsilon := \{\Omega \in \mathcal{D}_k : \text{there exists a surjective embedding} \\ \iota \text{ of } \Omega_0 \text{ to } \Omega \text{ such that } ||\iota - \iota_0||_{C^k} < \epsilon\}.$$

Then the sets $\mathcal{U}_\epsilon$ form a sub-basis for a topology on $\mathcal{D}_k$. This is called the $\mathbf{C^k}$ **topology on domains**.

Thm. (Krantz) [5] *Let* $(\Omega_j)_{j=1}^{\infty}$ *be a sequence of bounded domains in* $\mathbb{C}^n$ *that converges to a limit domain* Ω *in the topology of domains. Then for every point* $(z,w)\in\Omega\times\Omega$ *there exists* j_0 *such that for all* $j \geq j_0$ *there is* $(z,w)\in\Omega_j\times\Omega_j$ *and*

$$\mathbf{B}_\Omega(z,w) = \lim_{j\to\infty} \mathbf{B}_{\Omega_j}(z,w).$$

The analogous result (we consider C^2 topology of domains) is also true for the Szegö kernel [5].

Now let us define the **Poisson-Szegö kernel** and **Poisson-Bergman kernel** (also called the **Berezin kernel**). The construction of the Poisson-Szegö kernel is due to Hua [2]. He observed, that the Szegö kernel can be converted to get a kernel which also reproduces H^2, but which is positive (not only on the diagonal). Hua defined the Poisson-Szegö kernel as follows: $\mathcal{S}\colon \Omega\times\partial\Omega \to (0,+\infty)$ and

$$\mathcal{S}(z,w) := \frac{|\widetilde{\mathbf{S}}(z,w)|^2}{\mathbf{S}(z,z)}.$$

Now it is not difficult to prove the following theorem [5].

Thm. *For* $f\in\mathcal{O}(\Omega)\cap C(\overline{\Omega})$, *the equality*

$$f(z) = \int_{\partial\Omega} \mathcal{S}(z,w) f(w) d\sigma(w)$$

holds for all $z\in\Omega$.

Moreover, it is true [5], that this equality holds even for functions $f \in H^2(\Omega)$, if we assume that Ω is strictly pseudoconvex.

It is known [5,6], that in the case $n = 1$ we have $\mathbf{P} = \mathcal{S}$ for a simply connected domain, but this equality does not hold for $n > 1$.

Hua did not consider his construction for the Bergman kernel, but is is just as valid in this context. This ideas were developed by Berezin in the context of quantization on the Kähler manifolds [1]. If we define

$$\mathcal{B}(z,w) := \frac{|\mathbf{B}(z,w)|^2}{\mathbf{B}(z,z)},$$

then we have the following result.

Thm. *For $f \in \mathcal{O}(\Omega) \cap C(\overline{\Omega})$, the equality*

$$f(z) = \int_\Omega \mathcal{B}(z,w) f(w) dV(w)$$

holds for all $z \in \Omega$.

Moreover, it is true [5], that this equality holds even for functions $f \in A^2(\Omega)$, if we assume that Ω is pseudoconvex domain with C^∞ boundary.

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