# ON (WEAK) $m$-EXTREMALS AND $m$-GEODESICS

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For a domain $D \subset \mathbb{C}^n$ denote by $\mathcal{O}(\overline{\mathbb{D}}, D)$ the set of mappings $h$ such that there exists a neighborhood $U = U(h)$ of $\overline{\mathbb{D}}$ with $h \in \mathcal{O}(U, D)$. Let $m \geq 2$ and let $\lambda_1, \ldots, \lambda_m \in \mathbb{D}$ be pairwise distinct points. A holomorphic mapping $f : \mathbb{D} \longrightarrow D$ is called a *weak $m$-extremal* for $\lambda_1, \ldots, \lambda_m$ if there is no map $h \in \mathcal{O}(\overline{\mathbb{D}}, D)$ such that $h(\lambda_j) = f(\lambda_j)$, $j = 1, \ldots, m$. Naturally, weak $m$-extremality means weak $m$-extremality for some $\lambda_1, \ldots, \lambda_m$.

If the above condition is satisfied for any choice of $\lambda_1, \ldots, \lambda_m$, we say that $f$ is an *$m$-extremal.*

A holomorphic mapping $f : \mathbb{D} \longrightarrow D$ is said to be an *$m$-geodesic* if there exists $F \in \mathcal{O}(D, \mathbb{D})$ such that $F \circ f$ is a non-constant Blaschke product of degree at most $m - 1$.

Note that a holomorphic map is a weak 2-extremal (resp. a 2-geodesic) if and only if it is a Lempert extremal (resp. a geodesic).

In the talk the most important results of [1] are presented.

**Theorem.** *Any 3-extremal of the Euclidean ball is a 3-geodesic.*

Denote $\mathcal{E}(p) := \{z \in \mathbb{C}^n : |z_1|^{2p_1} + \ldots + |z_n|^{2p_n} < 1\}$.

**Proposition.** *Let $m \geq 3$ and $0 < a < 1$. Then the map*

$$f(\lambda) := (a\lambda^{m-2}, (1-a)\lambda^{m-1}), \quad \lambda \in \mathbb{D},$$

*is an $m$-extremal, but not an $m$-geodesic of $\mathcal{E}(1/2, 1/2)$.*

**Proposition.** *Let $p \in \{1, p_0\}^n$, where $p_0 \geq 1/2$, and let $f : \mathbb{D} \longrightarrow \mathcal{E}(p)$ be a holomorphic map. Then*

(a) *$f$ is a weak $m$-extremal if and only if it is an $m$-extremal.*

(b) *if $f_j \not\equiv 0$, $j = 1, \ldots, n$, it follows that $f$ is an $m$-extremal if and only if it is of the form*

$$f_j(\lambda) = a_j \prod_{k=1}^{m-1} \left(\frac{\lambda - \alpha_{kj}}{1 - \overline{\alpha}_{kj}\lambda}\right)^{r_{kj}} \left(\frac{1 - \overline{\alpha}_{kj}\lambda}{1 - \overline{\alpha}_{k0}\lambda}\right)^{1/p_j}, \quad j = 1, \ldots, n,$$

*where*

$$a_1, \ldots, a_n \in \mathbb{C}_*, \quad \alpha_{kj} \in \overline{\mathbb{D}}, \quad \alpha_{k0} \in \mathbb{D}, \quad r_{kj} \in \{0, 1\}, \quad r_{kj} = 1 \Longrightarrow \alpha_{kj} \in \mathbb{D},$$

$$\sum_{j=1}^{n} |a_j|^{2p_j} \prod_{k=1}^{m-1} (\lambda - \alpha_{kj})(1 - \overline{\alpha}_{kj}\lambda) = \prod_{k=1}^{m-1} (\lambda - \alpha_{k0})(1 - \overline{\alpha}_{k0}\lambda), \quad \lambda \in \mathbb{C},$$

*the case $r_{kj} = 0$, $k = 1, \ldots, m-1$, $j = 1, \ldots, n$ and*

*$\{\alpha_{kj} : k = 1, \ldots, m-1\} = \{\alpha_{k0} : k = 1, \ldots, m-1\}$ as multisets, $j = 1, \ldots, n$, is excluded.*

## References

[1] T. Warszawski, (*Weak*) *m-extremals and m-geodesics*, to appear in Complex Var. Elliptic Equ., arXiv: 1409.7585.