1. Results of J.E. Pascoe

Let Π denote the upper half plane in $\mathbb{C}$, i.e.

$$\Pi := \{z \in \mathbb{C} : \mathrm{Im} z > 0\}.$$

Theorem 1.1 ([Pas2014], Theorem 1.4)**.** *Let $f : \Pi^n \cup (-1, \varepsilon)^n \cup (-\varepsilon, 1)^n \to \overline{\Pi}$ (for fixed $\varepsilon > 0$) be:*

- *continuous on $\Pi^n \cup (-1, \varepsilon)^n \cup (-\varepsilon, 1)^n$,*
- *holomorphic on Π^n,*
- *real-valued on $(-1, \varepsilon)^n \cup (-\varepsilon, 1)^n$.*

Then for any $h \in \mathbb{C}^n$

$$\left| \frac{f^{(d)}(0)(h)}{d!} \right| \leq 6 \cdot 60^d \|h\|_\infty^d |f'(0)(1)|,$$

where $\|z\|_\infty := \max\{|z_j| : j = 1, \ldots, n\}$, $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

Proof of this theorem is based on two lemmas.

Lemmma 1.2 ([Pas2014], Lemma 2.1)**.** *Let $p(z)$ be a homogeneous polynomial of degree d in n variables such that $|p(z)| \leq 1$ for any $z \in [0, 1]^n$. Then for any $z \in \mathbb{C}^n$*

$$|p(z)| \leq (3\sqrt{2})^d \|z\|_\infty^d \frac{3^{9/4}}{64} \left((1 + \sqrt{2})^{d+1} + (1 - \sqrt{2})^{d+1} \right)^3 .$$

Lemmma 1.3 ([Pas2014], Lemma 3.1, [Pas2012], Theorem 4.3)**.** *Let $f : \Pi^n \cup (-1, \varepsilon)^n \cup (-\varepsilon, 1)^n \to \overline{\Pi}$ be as in Theorem 1.1. Then for any $h \in [0, 1]^n$*

$$\left| \frac{f^{(d)}(0)(h)}{d!} \right| \leq |f'(0)(1)|.$$

Proof of Theorem 1.1. For fixed $d \in \{1, 2, \ldots\}$ define $p(h) := \frac{f^{(d)}(0)(h)}{d! f'(0)(1)}$, $h \in \mathbb{C}^n$. Then p is a homogeneous polynomial of degree d. From Lemma 1.3, $|p(h)| \leq 1$ for any $h \in [0, 1]^n$ and we obtain the conclusion directly from Lemma 1.2. □

2. Similar results of J. Siciak

Let $P(\mathbb{C}^n)$ denote the set of all polynomials of n complex variables and let $H(\mathbb{C}^n)$ denote the set of homogeneous polynomials of n complex variables.

Definition 2.1. *For a compact set $E \subset \mathbb{C}^n$ we define* a polynomial extremal function

$$\Phi_E(z) := \sup\{|p(z)|^{\frac{1}{\deg p}} : \ p \in P(\mathbb{C}^n), \ \deg p \geq 1, \ \|p\|_E \leq 1\}, \ z \in \mathbb{C}^n,$$

and a homogeneous extremal function

$$\Psi_E(z) := \sup\{|p(z)|^{\frac{1}{\deg p}} : \ p \in H(\mathbb{C}^n), \ \deg p \geq 1, \ \|p\|_E \leq 1\}, \ z \in \mathbb{C}^n,$$

where $\|p\|_E := \sup_{z \in E} |p(z)|$.

Theorem 2.2 (see [Sic1981])**.** *Let $E \subset F$ be convex subsets of $\mathbb{C}^n$. Then $\Phi_F \leq \Phi_E$ and $\Psi_F \leq \Psi_E$.*

Theorem 2.3.

$$\Phi_{[-1,1]}(z) = |z + \sqrt{z^2 - 1}|.$$

Theorem 2.4 (Product property, see [Sic1981])**.** *Let $E_j \subset \mathbb{C}^{n_j}$ be a compact set, $j = 1, \ldots, N$. Then*

$$\Phi_{E_1 \times \ldots \times E_N}(z) = \max\{\Phi_{E_1}(z_1), \ldots, \Phi_{E_N}(z_N)\}.$$

Theorem 2.5 (see [Sic1981])**.** *Let $E \subset \mathbb{C}^n$ be a compact circular set. Then $\Phi_E(z) = \max\{1, \Psi_E(z)\}$, $z \in \mathbb{C}^n$. In particular, if $\mathbb{B}_n$ is a closed unit ball in $\mathbb{C}^n$ with respect to a norm q, then $\Psi_{\mathbb{B}_n}(z) = q(z)$.*

Theorem 2.6 (see [Sic1961], [Lun1985])**.** *Let $B \subset \mathbb{C}^n$ be a closed unit Euclidean real ball, i.e.*

$$B = \{z \in \mathbb{C}^n : \ \mathrm{Im} z_1 = \ldots = \mathrm{Im} z_n = 0, \ (\mathrm{Re} z_1)^2 + \ldots + (\mathrm{Re} z_n)^2 \leq 1\}.$$

Then

$$\Phi_B(z) = \sqrt{\|z\|^2 + |z^2 - 1| + \sqrt{(\|z\|^2 + |z^2 - 1|)^2 - 1}},$$

$$\Psi_B(z) = L(z),$$

where $L(z)$ denotes the Lie norm

$$L(z) = \left(\|z\|^2 + \sqrt{\|z\|^4 - |z^2|^2} \right)^{\frac{1}{2}}, \; z \in \mathbb{C}^n, \; z^2 := z_1^2 + \ldots + z_n^2$$

and $\|z\|$ denotes the Euclidean norm.

Using extremal functions the following results, similar to Lemma 1.2, can be obtain.

Lemmma 2.7. *Let $p(z)$ be a homogeneous polynomial of degree d in n complex variables such that $\|p\|_{[-1,1]^n} \leq 1$. Then for any $z \in \mathbb{C}^n$*

$$|p(z)| \leq \left(\frac{\sqrt{2}+1}{\sqrt{2}} \|z\| \right)^d.$$

Proof. Let $K := [-1,1]^n$ and fix $p \in H(\mathbb{C}^n)$, $\deg p = d$, such that $\|p\|_K \leq 1$. Observe that $B \subset K$. From the definition and properties of homogeneous extremal function we get for any $z \in \mathbb{C}^n$

$$|p(z)| \leq (\Psi_K(z))^d \leq (\Psi_B(z))^d = \left(\sqrt{\|z\|^2 + \sqrt{\|z\|^4 - |z^2|^2}} \right)^d$$

$$= \left(\frac{\sqrt{\|z\|^2 + |z^2|} + \sqrt{\|z\|^2 - |z^2|}}{\sqrt{2}} \right)^d \leq \left(\frac{\sqrt{2}+1}{\sqrt{2}} \|z\| \right)^d.$$

□

Lemmma 2.8. *Let p be a polynomial (not necessarily homogeneous) of degree d in n complex variables such that $\|p\|_{[-1,1]^n} \leq 1$. Then for any $z \in \mathbb{C}^n$*

$$|p(z)| \leq \left(\sqrt{6} \max\{1, \|z\|\} \right)^d.$$

Proof. Let $K := [-1,1]^n$ and fix $p \in P(\mathbb{C}^n)$, $\deg p = d$, such that $\|p\|_K \leq 1$. Then $\|p\|_B \leq 1$ and from the definition of polynomial extremal function we get for any $z \in \mathbb{C}^n$

$$|p(z)| \leq (\Phi_B(z))^d = \left(\sqrt{\|z\|^2 + |z^2 - 1| + \sqrt{(\|z\|^2 + |z^2 - 1|)^2 - 1}} \right)^d$$

$$\leq \left(\sqrt{2\|z\|^2 + 1 + \sqrt{(2\|z\|^2 + 1)^2 - 1}} \right)^d \leq \left(\sqrt{4\|z\|^2 + 2} \right)^d \leq \left(\sqrt{6} \max\{1, \|z\|\} \right)^d.$$

□

Lemmma 2.9. *Let p be a homogeneous polynomial of degree d in n complex variables such that $\|p\|_{[0,1]^n} \leq 1$. Then for any $z \in \mathbb{C}^n$*

$$|p(z)| \leq \left(3\sqrt{6} \|z\| \right)^d.$$

Proof. Fix p as in the assumptions and define $q(z_2, \ldots, z_n) := p(1, \frac{z_2+1}{2}, \ldots, \frac{z_n+1}{2})$. Then $\deg q \leq \deg p$ and for any $(z_2, \ldots, z_n) \in [-1,1]^n$ we have $(1, \frac{z_2+1}{2}, \ldots, \frac{z_n+1}{2}) \in [0,1]^n$, thus $\|q\|_{[-1,1]^n} \leq 1$. Hence, from Lemma 2.8,

$$|q(z_2, \ldots, z_n)| \leq \left(\sqrt{6} \max\{1, \|z'\|\} \right)^{\deg q},$$

where $z' = (z_2, \ldots, z_n)$.

Observe, that $p(z_1, \ldots, z_n) = z_1^d p(1, \frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1})$. Thus, for any $z \in \mathbb{C}^n$,

$$|p(z)| = |z_1|^d |q(\frac{2z_2 - z_1}{z_1}, \ldots, \frac{2z_n - z_1}{z_1})| \leq |z_1|^d \left(\sqrt{6} \max\{1, \frac{\|w'\|}{|z_1|}\} \right)^{\deg q},$$

where $w' = (2z_2 - z_1, \ldots, 2z_n - z_1)$.

Hence

$$|p(z)| \leq |z_1|^d \left(\sqrt{6} \max\{1, \frac{\|w'\|}{|z_1|}\} \right)^d \leq \left(\sqrt{6} \max\{|z_1|, \|w'\|\} \right)^d$$

$$\leq \left(\sqrt{6} \max\{\|z\|, \|2z - z_1\|\} \right)^d \leq \left(3\sqrt{6} \|z\| \right)^d.$$

□

Lemmma 2.10. *Let p be a polynomial (not necessarily homogeneous) of degree d in n complex variables such that $\|p\|_{[-1,1]^n} \leq 1$. Then for any $z \in \mathbb{C}^n$*

$$|p(z)| \leq \Big((1+\sqrt{2}) \max\{1, \|z\|_\infty\}\Big)^d.$$

Proof. Fix p as in the assumptions. Then, from Theorems 2.3 and 2.4, for any $z \in \mathbb{C}^n$

$$|p(z)| \leq \big(\Phi_{[-1,1]^n}(z)\big)^d = \Big(\max\{|z_j + \sqrt{z_j^2 - 1}|,\ j = 1, \dots, n\}\Big)^d \leq ((1+\sqrt{2}) \max\{1, \|z\|_\infty\})^d.$$

□

Lemmma 2.11. *Let p be a homogeneous polynomial of degree d in n complex variables such that $\|p\|_{[0,1]^n} \leq 1$. Then for any $z \in \mathbb{C}^n$*

$$|p(z)| \leq \Big(3(1+\sqrt{2})\|z\|_\infty\Big)^d.$$

Proof. We use standard dehomogenization/rehomogenization method. Fix p as in the assumptions and define $q(z_2, \dots, z_n) := p(1, \frac{z_2+1}{2}, \dots, \frac{z_n+1}{2})$ (dehomogenization). Then $\deg q \leq \deg p$ and $\|q\|_{[-1,1]^n} \leq 1$. Thus, from Lemma 2.10,

$$|q(z_2, \dots, z_n)| \leq \Big((1+\sqrt{2}) \max\{1, |z_j|, j = 2, \dots, n\}\Big)^{\deg q}.$$

Observe, that $p(z_1, \dots, z_n) = z_1^d p(1, \frac{z_2}{z_1}, \dots, \frac{z_n}{z_1})$ (rehomogenization). Hence, for any $z \in \mathbb{C}^n$,

$$\begin{aligned} |p(z)| = |z_1|^d |q(\frac{2z_2 - z_1}{z_1}, \dots, \frac{2z_n - z_1}{z_1})| &\leq |z_1|^d \left((1+\sqrt{2}) \max\{1, |\frac{2z_2 - z_1}{z_1}|, \dots, |\frac{2z_n - z_1}{z_1}|\}\right)^{\deg q} \\ &\leq |z_1|^d \left((1+\sqrt{2}) \max\{1, |\frac{2z_2 - z_1}{z_1}|, \dots, |\frac{2z_n - z_1}{z_1}|\}\right)^d \\ &\leq \Big((1+\sqrt{2}) \max\{|z_1|, |2z_2 - z_1|, \dots, |2z_n - z_1|\}\Big)^d \leq \Big(3(1+\sqrt{2})\|z\|_\infty\Big)^d. \end{aligned}$$

□

References

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