CHARACTERIZATIONS OF BOUNDARY PLURIPOLAR HULLS

Abstract. We present some basic properties of the boundary relative extremal function. On using Levenberg-Poletsky's method we give a characterization of boundary pluripolar hull and by Zeriahi's approach we prove the completeness of every boundary pluripolar set that is F_σ and G_δ.

1. Introduction

D will denote a bounded domain in $\mathbb{C}^n$, $PSH(D)$ the family of all plurisubharmonic functions on D and A a subset in the boundary of D. For any function $u : D \to \mathbb{R} \cup \{-\infty\}$ and $x \in \overline{D}$ we set

$$u^*(x) = \limsup_{z \to x, z \in D} u(z) = \lim_{r \to 0} \sup_{B(x,r) \cap D} u$$

$\mathbb{D}$ will be the unit disk in $\mathbb{C}$, $\mathbb{T}$ the unit circle and σ the arc length measure on $\mathbb{T}$. $\mathscr{O}(\mathbb{D}, D) \cap C(\overline{\mathbb{D}}, D)$ is the set of holomorphic maps $f : \mathbb{D} \to D$ continuous on $\overline{\mathbb{D}}$. We will always consider A as a subset of ∂D. We define for $z \in D$,

$$\omega(z, A, D) = \sup\{u(z) : u \in PSH(D), u < 0, u^* \leq -1 \text{ on } A\}.$$

We will call $\omega^*(., A, D)$ the *boundary relative extremal function*.

For $A \subset \partial D$, it can happen that any $u \in PSH(D)$ such that $u^*|A = -\infty$ takes automatically $-\infty$ on a bigger set in $\overline{D}$. For instance, set $D = \{(z_1, z_2) \in \mathbb{C}^2, |z_1^2| + |z_2^2| < 1\}$. Let $A_1 \subset \mathbb{T}$ be the closure of a half-circle. Set $A = A_1 \times \{0\}$. Any $u \in PSH(D)$ such that $u^* \equiv -\infty$ on A is identically $-\infty$ in $\{z \in \mathbb{C}, |z| < 1\} \times \{0\}$. We will use $\omega(., A, D)$ to describe this phenomenon of propagation.

Definition 1. We say that a subset $A \in \partial D$ is a b-pluripolar set if there exists a $u \in PSH(D)$, $u \leq 0$, $u \not\equiv -\infty$, such that $u^* = -\infty$ on A.

Let $A \subset \partial D$ be b-pluripolar, the set

$$\{z \in \overline{D}, \quad u^*(z) = -\infty, \text{ for all } u \in PSH(D) \text{ with } u \not\equiv -\infty, \quad u < 0, \quad u^* = -\infty \text{ on } A\}$$

will be called the $b-$*pluripolar hull* of A and will be denoted by $\hat{A}$.

Theorem 1. *Let $D \subset \mathbb{C}^n$ be B-regular and $A \subset \partial D$ be an F_σ set b-pluripolar. Then*

$$\hat{A} = A \cup \{z \in D, \quad \omega(z, A, D) < 0\}.$$

Definition 2. We say that a subset $A \in \partial D$ is complete b-pluripolar if there exists a $u \in PSH(D)$, $u < 0$, $\quad u \not\equiv -\infty$, such that $\{z \in \partial D, \quad u^*(z) = -\infty\} = A$.

Theorem 2. *Let $A \subset \partial D$ be b-pluripolar, F an F_σ set, G a G_δ set on ∂D such that $F \subset A \subset \hat{A} \cap \partial D \subset G$. Then there is $\tilde{E} \subset \partial D$ complete b-pluripolar so that $F \subset \tilde{E} \subset G$.*

Remark 3. Every b-pluripolar set that is on the same time F_σ and G_δ is complete b-pluripolar. In particular, every closed b-pluripolar set is complete b-pluripolar.

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