Let X be a topological Hausdorff space. We define an equivalence relation on X^k by $(x_1, \ldots, x_k) \sim (y_1, \ldots, y_k) :\Longleftrightarrow (y_1, \ldots, y_k)$ is a reordering of $(x_1, \ldots, x_k)$. Then we define the k-th symmetric product of X: $\overleftrightarrow{X^k} := X^k/\sim$. In the case $k = 1$, we get $\overleftrightarrow{X^1} = X$. Now, we define the projection $\pi : X^k \longrightarrow \overleftrightarrow{X^k}$, $\pi(x) := [x]$. We put $[x_1, \ldots, x_k] := [(x_1, \ldots, x_k)]$, $\{[x_1, \ldots, x_k]\} := \{x_1, \ldots, x_k\}$. If $x_j \neq x_t$ for $j \neq t$, then we define $[[x_1, \ldots, x_k]] := [x_1, \ldots, x_k]$. Moreover, we put

$$[x_1 : \mu_1, \ldots, x_\ell : \mu_\ell] := [\overbrace{x_1, \ldots, x_1}^{\mu_1\text{-times}}, \ldots, \overbrace{x_\ell, \ldots, x_\ell}^{\mu_\ell\text{-times}}],$$

provided that $x_j \neq x_t$ for $j \neq t$, $\mu_1, \ldots, \mu_\ell \in \mathbb{N}$, $\mu_1 + \cdots + \mu_\ell = k$. We define

$$[A_1, \ldots, A_k] := \Big\{ [x_1, \ldots, x_k] : x_i \in A_i, \quad i = 1, \ldots, k \Big\},$$

The topology on $\overleftrightarrow{X^k}$ is defined by the basis

$$[U_1, \ldots, U_m], \quad U_i \text{ is open in } X, \quad i = 1, \ldots, k.$$

Observe that π is continuous, open, and $\overleftrightarrow{X^k}$ is Hausdorff.

Definition 0.1. Let Y be Hausdorff topological space and let $F : X \longrightarrow \overleftrightarrow{Y^n}$ be continuous. Then we put

$$X_F^{(k)} := \{x \in X : \#\{F(x)\} = k\},$$
$$\chi_F := \max\{k : X_F^{(k)} \neq \varnothing\}, \quad X_F := X_F^{(\chi_F)}.$$

Note that X_F is open.

Proposition 0.2. *Let F be as above. Suppose that*

$$a \in X_F, \quad F(a) = [b_1 : \mu_1, \ldots, b_k : \mu_k], \quad k := \chi_F.$$

Then there is a neighborhood $U \subset X_F$ of a and there are uniquely defined continuous functions $f_i : U \longrightarrow Y$, $i = 1, \ldots, k$, such that

$$F(x) = [f_1(x) : \mu_1, \ldots, f_k(x) : \mu_k], \quad x \in U.$$

In the above situation, we will write $F = \mu_1 f_1 \oplus \cdots \oplus \mu_k f_k$ on U.

Proposition 0.3. *Let $F : X^k \longrightarrow Y$. Then there exists a continuous function $\overleftrightarrow{F} : \overleftrightarrow{X^k} \longrightarrow Y$ such that $F = \overleftrightarrow{F} \circ \pi$ if and only if F is symmetric.*

Definition 0.4. Let M, N be complex manifolds and let M be connected. We say a continuous function $F : M \longrightarrow \overleftrightarrow{N^n}$ is *holomorphic on* M ($F \in \mathcal{O}(M, \overleftrightarrow{N^n})$), if:

- $M \setminus M_F$ is thin, i.e. every point $x_0 \in M \setminus M_F$ has open connected neighborhood $V \subset M$ and a function $\varphi \in \mathcal{O}(V)$, $\varphi \not\equiv 0$, such that $(M \setminus M_F) \cap V \subset \varphi^{-1}(0)$,
- for every $a \in M_F$, if $F = \mu_1 f_1 \oplus \cdots \oplus \mu_k f_k$ on V as in Proposition 0.2, then $f_1, \ldots, f_k \in \mathcal{O}(V)$.

If M is disconnected, then we say that F is *holomorphic on* M, if $F|_C \in \mathcal{O}(C, \overleftrightarrow{N^n})$ for any connected component $C \subset M$.

Proposition 0.5. *Let M, N, K be a complex manifolds and let $f \in \mathcal{O}(M, N)$, $g \in \mathcal{O}(N, \overleftrightarrow{K^n})$. Assume that $f(M) \cap N_g \neq \varnothing$. Then $g \circ f \in \mathcal{O}(M_{g \circ f}, \overleftrightarrow{K^n})$.*

Proposition 0.6. *Let* $f \in \mathcal{O}(M, \overleftrightarrow{N^n})$ *and* $g \in \mathcal{O}(N^n, K)$ *be symmetric. Then* $\overleftrightarrow{g} \circ f \in \mathcal{O}(M, K)$.

Definition 0.7. Let M be an analytic submanifold of a manifold X. Let $U \subset X$ be a domain such that $U \cap M \neq \varnothing$. We say a holomorphic function

$$\Delta : U \longrightarrow \overleftrightarrow{(M \times \mathbb{C})^n}$$

is a *holomorphic multivalued projection* $U \longrightarrow M$, if for any $x \in U \cap M$ such that $\Delta(x) = [(x_1, z_1), \ldots, (x_n, z_n)]$ we have $x_{j_0} = x$ for some $j_0 \in \{1, \ldots, n\}$ and $z_j = 0$ for any $j \in \{1, 2, \ldots, n\} \setminus \{j_0\}$.

Let $\mathcal{P}$ denote set of all holomorphic multivalued projections $U \longrightarrow M$. Then we define the map

$$\Xi : (U \cap M) \times \mathcal{P} \longrightarrow \mathbb{C}, \quad \Xi(x, \Delta) := z_{j_0}.$$

Observe that Ξ is well defined.

Definition 0.8. We say $\Pi = (\Delta_j)_{j=1}^k$ is an *system of holomorphic multivalued projections* $U \longrightarrow M$, if $\Delta_j : U \longrightarrow \overleftrightarrow{(M \times \mathbb{C})^{n_j}}$, $j = 1, \ldots, k$, are holomorphic multivalued projections and $\sum_{j=1}^k \Xi(x, \Delta_j) = 1$ for any $x \in U \cap M$.

Theorem 0.9. *Assume that there exists a system* Π *of holomorphic multivalued projections on* U. *Then exists a linear continuous operator*

$$L_\Pi : \mathcal{O}(M) \longrightarrow \mathcal{O}(U)$$

such that $L_\Pi(u)(x) = u(x)$ *for* $x \in U \cap M$.

Theorem 0.10. *Let* M *be an analytic submanifold of a Stein manifold* X. *Let* U *be a relatively compact domain of* X *such that* $U \cap M \neq \varnothing$. *Then there exists a system of multivalued holomorphic projections* $U \longrightarrow M$.

Definition 0.11. Let $f \in \mathcal{O}(X, \mathbb{C}^k)$. We say that a set $P \subset P_0 := M \cap f^{-1}(\mathbb{D}^k)$ is an *analytic polyhedron* in M ($P \in \mathcal{P}(M, k, f)$) if $P \subset\subset M$ and P is the union of a family of connected components of P_0.

We say that an analytic polyhedron $P \in \mathcal{P}(M, k, f)$ is *special* if $d = k$.

Theorem 0.12 (cf. [2])**.** *Assume that* $P \in \mathcal{P}(M, k, f)$, $S \subset P$, $T \subset f^{-1}(\mathbb{D}^k)$ *are compact. Then there exists a special analytic polyhedron* $Q \in \mathcal{P}(M, d, g)$ *such that* $S \subset Q \subset P$ *and* $g(T) \subset \mathbb{D}^d$.

Theorem 0.13 (cf. [2])**.** *Assume that* X *is Stein,* $T \subset X$ *is compact, and* U *is an open neighborhood of* T *such that* $(U \setminus T) \cap \hat{T}_{\mathcal{O}(X)} = \varnothing$. *Let* $\mathcal{A} := \mathrm{cl}_{\mathcal{C}(T)}(\mathcal{O}(U)|_T)$. *Then* $\mathrm{Spec}(\mathcal{A}) = T$, *i.e every non-zero character (homomorphism)* $\xi : \mathcal{A} \longrightarrow \mathbb{C}$ *is an evaluation (i.e. there exists an* $x_0 \in T$ *such that* $\xi(f) = f(x_0)$ *for every* $f \in \mathcal{A}$*).*

Consequently (cf. [1]*, Chapter I, Section II, Corollary 10), if* $w_1, \ldots, w_m \in \mathcal{A}$ *have no common zeros on* T, *then there exist* $c_1, \ldots, c_m \in \mathcal{A}$ *such that* $c_1 w_1 + \cdots + c_m w_m = 1$.

Theorem 0.14 (cf. [2])**.** *Assume that* $P \in \mathcal{P}(M, d, f)$ *is special. Then there exist a* $k \in \mathbb{N}$ *and a holomorphic mapping* $\omega : \mathbb{D}^d \longrightarrow \overleftrightarrow{P^k}$ *such that:*

- $f^{-1}(z) \cap P = \{\omega(z)\}$, $z \in \mathbb{D}^d$,
- $\#\{\omega(z)\} = k$ *for* $z \in \mathbb{D}^d \setminus \Sigma'$, *where* Σ' *is a proper analytic set.*

Proposition 0.15. *Let* ω*,* f*,* X*,*P *be as above. Additionally assume that* $f(U) \subset \mathbb{D}^d$*, where* $U \subset X$ *is a domain and* $U \cap P \neq \varnothing$*. Then* $\omega \circ f|_U \in \mathcal{O}(U, \overleftrightarrow{P^k})$*.*

Proposition 0.16. *Let* ω*,* f*,* X*,*P *be as above. Then* $\omega \circ f|_P \in \mathcal{O}(P, \overleftrightarrow{P^k})$*.*

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