A counterexample to a theorem of Bremermann on Shilov boundaries — revisited

Marek Jarnicki, Peter Pflug (November 9, 2015)

For a bounded domain $D \subset \mathbb{C}^n$ let $\mathcal{A}(D)$ (resp. $\mathcal{O}(\overline{D})$) denote the space of all continuous functions $f : \overline{D} \longrightarrow \mathbb{C}$ such that $f|_D$ is holomorphic (resp. f extends holomorphically to a neighborhood of $\overline{D}$). Let $\partial_S D$ (resp. $\partial_B D$) be the *Shilov* (resp. *Bergman*) *boundary* of D, i.e. the minimal compact set $K \subset \overline{D}$ such that $\max_K |f| = \max_{\overline{D}} |f|$ for every $f \in \mathcal{A}(D)$ (resp. $f \in \mathcal{O}(\overline{D})$). Obviously, $\mathcal{O}(\overline{D}) \subset \mathcal{A}(D)$ and hence $\partial_B D \subset \partial_S D \subset \partial D$. Notice that, in general, $\partial_B D \subsetneq \partial_S D$, e.g. for the domain $D := \{(z,w) \in \mathbb{C}^2 : 0 < |z| < 1,\ |w| < |z|^{-\log|z|}\}$.

The algebra $\mathcal{A}(D)$ (resp. $\mathcal{B}(D) :=$ the uniform closure in $\mathcal{A}(D)$ of $\mathcal{O}(\overline{D})$) endowed with the supremum norm is a Banach algebra. A point $a \in \overline{D}$ is called a peak point for $\mathcal{A}(D)$ (resp. $\mathcal{B}(D)$) if there is an $f \in \mathcal{A}(D)$ (resp. $\mathcal{B}(D)$) with $f(a) = 1$ and $|f(z)| < 1$ for all $z \in \overline{D} \setminus \{a\}$; f is called an associated *peak function*. It is known the peak points of $\mathcal{A}(D)$ (resp. $\mathcal{B}(D)$) are dense in $\partial_S D$ (resp. $\partial_B(D)$)). Assume that the envelope of holomorphy $\widetilde{D}$ of D is univalent. Note that $\partial_S \widetilde{D} \subset \partial_S D$ and $\partial_B \widetilde{D} \subset \partial_B D$. In the paper:

[JP] M. Jarnicki, P. Pflug, *A counterexample to a theorem of Bremermann on Shilov boundaries*, Proc. Amer. Math. Soc. 143 (2015), 1675–1677,

we were interested in answering whether $\partial_S D = \partial_S \widetilde{D}$ (resp. $\partial_B D = \partial_B \widetilde{D}$). We studied the following bounded Hartogs domain $D \subset \mathbb{C}^2$:

$$D := \{(re^{i\varphi}, w) \in \mathbb{C}^2 : \tfrac{1}{2} < r < 1,\ \varphi \in (0, 2\pi),\ \begin{cases} 0 < \varphi \le \frac{\pi}{2} & \Longrightarrow |w| < 1 \\ \frac{\pi}{2} < \varphi < \frac{3\pi}{2} & \Longrightarrow |w| < 3 \\ \frac{3\pi}{2} \le \varphi < 2\pi & \Longrightarrow 2 < |w| < 3 \end{cases} \};$$

it is known that D has a univalent envelope of holomorphy $\widetilde{D}$. The main result of [JP] is the following theorem.

Theorem 1. *$\partial_S \widetilde{D} \subsetneq \partial_S D$, $\partial_B \widetilde{D} \subsetneq \partial_B D$, and $\mathcal{O}(\overline{D}) \setminus \mathcal{A}(\widetilde{D})|_{\overline{D}} \neq \varnothing$.*

The proof had the following two parts:

(1) $\partial_S \widetilde{D} \cap (I \times \mathbb{D}(3)) = \varnothing$, where $I := [\frac{1}{2}, 1]$, $\mathbb{D}$ is the unit disc, and $\mathbb{D}(r) := r\mathbb{D}$.

(2) There exists a function $h \in \mathcal{O}(\overline{D})$ (effectively given) such that

$$h(x,w) = \begin{cases} e^{-2\pi + i\log x}, & \text{if } (x,w) \in I \times \overline{\mathbb{A}}(2,3) \\ e^{i\log x}, & \text{if } (x,w) \in I \times \overline{\mathbb{D}} \end{cases}$$

and $|h| < 1$ on the remaining part of $\overline{D}$, where $\mathbb{A}(r_-, r_+) := \mathbb{D}(r_+) \setminus \overline{\mathbb{D}}(r_-)$.

Unfortunately, the proof of (1) contains a gap. The aim of the present note is to close the above gap and to prove some new results related to the Shilov and Bergman boundaries of D and $\widetilde{D}$.

Let

$$A := \{z \in \mathbb{C} : \tfrac{1}{2} < |z| < 1\}, \quad I_0 := (\tfrac{1}{2}, 1), \quad A_0 := A \setminus I_0.$$

By the Cauchy integral formula each function $f \in \mathcal{A}(D)$ extends holomorphically to the domain

$$G = \{(z,w) \in A_0 \times \mathbb{C} : |w|e^{V(z)} < 1\},$$

where $V(re^{i\varphi}) := \begin{cases} 0, & \text{if } 0 < \varphi \le \frac{\pi}{2} \\ -\log 3, & \text{if } \frac{\pi}{2} < \varphi < 2\pi \end{cases}$. Hence the envelope of holomorphy $\widetilde{D}$ is univalent and

$$\widetilde{D} = \widetilde{G} = \{(z,w) \in A_0 \times \mathbb{C} : |w|e^{\widetilde{V}(z)} < 1\}, \text{ where } \widetilde{V}(z) := \sup\{u \in \mathcal{SH}(A_0) : u \le V\}.$$

Notice that, by the maximum principle for subharmonic functions, we have $\widetilde{V}(z) < 0$, $z \in A_0$. Thus, $\partial D \cap (U \times \mathbb{D}(3)) \subset \widetilde{D}$, where $U := \{re^{i\varphi} : r \in I_0,\ 0 < \varphi < \frac{\pi}{2}\}$. Hence $\partial \widetilde{D} \cap (U \times \mathbb{D}(3))$ does not contain points of $\partial_S \widetilde{D}$.

We are going to prove the following theorem.

Theorem 2. (a) $\partial_S \widetilde{D} \cap (I_0 \times \mathbb{D}(3)) = \varnothing$.

(b) *For any* $a \in I$ *there exists a* $g = g_a \in \mathcal{O}(\overline{D})$ *such that* $g(a,w) = 1$ *for all* $w \in \overline{\mathbb{D}}$*, and* $|g| < 1$ *on* $\overline{D} \setminus (\{a\} \times \overline{\mathbb{D}})$*. In particular,* $\partial_B D \cap (\{a\} \times \overline{\mathbb{D}}) \neq \varnothing$*.*
(c) $\{a\} \times \mathbb{T} \subset \partial_B D$ *for every* $a \in I_0$*, where* $\mathbb{T} := \partial\mathbb{D}$*. Therefore,* $I \times \mathbb{T} \subset \partial_B D$*.*
(d) *Let* $M_0 := (iI_0) \times (3\mathbb{T})$*. Then* $\partial_B D \setminus M_0 = \partial_S D \setminus M_0$

$$= \Big(\{re^{i\varphi} : r \in \{\tfrac{1}{2}, 1\},\ \tfrac{\pi}{2} \le \varphi \le 2\pi\} \times (3\mathbb{T})\Big) \cup \Big(I_0 \times (3\mathbb{T})\Big) \cup \Big(I_0 \times \mathbb{T}\Big) \cup \Big(\{re^{i\varphi} : r \in \{\tfrac{1}{2}, 1\},\ 0 \le \varphi \le \tfrac{\pi}{2}\} \times \mathbb{T}\Big)$$

$=: M_1 \cup M_2 \cup M_3 \cup M_4$.

Remark 3. (i) Observe that (a) and (b) close gaps in our former proof.

Indeed, we get $\varnothing \neq I_0 \times \mathbb{T} \subset \partial_B D \setminus \partial_S \widetilde{D} \subset (\partial_S D \setminus \partial_S \widetilde{D}) \cap (\partial_B D \setminus \partial_B \widetilde{D})$. Hence $\partial_S \widetilde{D} \subsetneq \partial_S D$ and $\partial_B \widetilde{D} \subsetneq \partial_B D$.

Moreover, if $a \in I_0$, then $g_a \in \mathcal{O}(\overline{D}) \setminus \mathcal{A}(\widetilde{D})|_{\overline{D}}$.

(ii) It seems to be an *open problem* whether $M_0 \subset \partial_S D$ (resp. $M_0 \subset \partial_B D$).

Proof of Theorem 2. First, let us make the following elementary observation.

(*) Let Σ be an open subset of the boundary $\partial\Omega$ of a bounded domain $\Omega \subset \mathbb{C}^n$. Suppose that $\max_{\partial\Omega} |f| = \max_{\partial\Omega\setminus\Sigma} |f|$ for every $f \in \mathcal{A}(\Omega)$ (resp. $\mathcal{O}(\overline{\Omega})$). Then $\partial_S \Omega \cap \Sigma = \varnothing$ (resp. $\partial_B \Omega \cap \Sigma = \varnothing$).

(a) For all $a \in I_0$ and $f \in \mathcal{A}(\widetilde{D})$ the function $f(a, \cdot)$ extends holomorphically to $\mathbb{D}(3)$.

Indeed, we may define $\widehat{f}(z,w) := \frac{1}{2\pi i} \int_{|\zeta|=5/2} \frac{f(z,\zeta)}{\zeta - w} d\zeta$, $z \in \mathbb{A}(\frac{1}{2}, 1)$, $\frac{\pi}{2} < \arg z \le 2\pi$, $|w| < \frac{5}{2}$. Then $\widehat{f}$ is holomorphic and coincide with f when $\frac{\pi}{2} < \arg z < \pi$. Hence using identity theorem we see that $f = \widehat{f}$ on their common domain of definition. Using continuity of f we get the claimed extension of $f(a, \cdot)$.

In particular, $\max_{\{a\}\times\overline{\mathbb{A}}(1,3)} |f(a,\cdot)| = \max_{\{a\}\times 3\mathbb{T}} |f(a,\cdot)|$. Hence, by (*) with $\Omega := \widetilde{D}$ and $\Sigma := I_0 \times \mathbb{A}(1,3)$, we conclude that $\partial_S \widetilde{D} \cap (I_0 \times \mathbb{A}(1,3)) = \varnothing$. The same argument shows that $\partial_S \widetilde{D} \cap (I_0 \times \mathbb{D}) = \varnothing$.

Suppose that $(z_0, w_0) \in (I_0 \times \mathbb{T}) \cap \partial_S \widetilde{D}$. Then there is a peak point (z_1, w_1) nearby. Let $f \in \mathcal{A}(\widetilde{D})$ be a function peaking there. The maximum principle excludes the situation where $z_1 \in I_0$. Thus $z_1 \in U$, but we already know that $\partial_S \widetilde{D} \cap (U \times \mathbb{D}(3)) = \varnothing$, so it is impossible.

Finally, $\partial_S \widetilde{D} \cap (I_0 \times \mathbb{D}(3)) = \varnothing$.

(b) Fix an $a \in I$ and let h be as in (2), $w_0 := e^{i \log a} \in \mathbb{T}$. Define $\varphi(w) := \frac{1}{2w_0}(w + w_0)$, $g := \varphi \circ h$. It is obvious that $g \in \mathcal{O}(\overline{D})$, $g(a,w) = 1$ for all $w \in \overline{\mathbb{D}}$, and $|g| < 1$ on $\overline{D} \setminus (\{a\} \times \overline{\mathbb{D}})$.

(c) Using (*) we have $\partial_B D \cap (I_0 \times \mathbb{D}) = \varnothing$. Hence, by (b), $\partial_B D \cap (\{a\} \times \mathbb{T}) \neq \varnothing$ for every $a \in I_0$. Now using rotational invariance in the second variable of $\partial_B D$ leads to $\{a\} \times \mathbb{T} \subset \partial_B D$ for all $a \in I_0$.

(d) Notice that also $\partial_S D$ is invariant under rotations of the second variable.

- Every point from $\partial A \times 3\mathbb{T}$ is a peak point for $\mathcal{O}(\overline{A \times \mathbb{D}(3)})$.

Indeed, fix a point $(a,b) \in \partial A \times 3\mathbb{T}$. Then a is a peak point for $\mathcal{O}(\overline{A})$ and b is a peak point for $\mathcal{O}(\overline{\mathbb{D}(3)})$. So it suffices to take the product of the corresponding peak functions to see that (a,b) is a peak point for $\mathcal{O}(\overline{A \times \mathbb{D}(3)})$.

Thus $M_1 \subset \partial_B D \subset \partial_S D$.

- Consider the holomorphic function $\overline{D} \ni (z,w) \overset{\Phi}{\longmapsto} \mathrm{Log}(z) \in R := [-\log 2, 0] \times [0, 2\pi]$, where Log is a branch of logarithm with $\mathrm{Log}(-1) = \pi$. Note that $\Phi \in \mathcal{O}(\overline{D})$. For every $a \in I_0$ we have $\Phi(a,w) \in (-\log 2, 0) \times \{2\pi\} \in \partial R$ whenever $w \in \overline{\mathbb{A}}(2,3)$. It is clear that there exists a function $\psi_a \in \mathcal{O}(\overline{R})$ such that $\psi_a(\Phi(a,w)) = 1$, $w \in \overline{\mathbb{A}}(2,3)$, and $|\psi_a| < 1$ on $\overline{R} \setminus \{\Phi(a,w)\}$. Then the function $\overline{D} \ni (z,w) \overset{f_a}{\longmapsto} \psi_a(\Phi(z,w))$ may be considered as a function of class $\mathcal{O}(\overline{D})$. Observe that $f_a(a,w) = 1$ for all $w \in \overline{\mathbb{A}}(2,3)$, and $|f_a| < 1$ on $\overline{D} \setminus (\{a\} \times \overline{\mathbb{A}}(2,3))$. Fix a $b \in 3\mathbb{T}$ and define $g(z,w) := f_a(z,w)\frac{1+w/b}{2}$. Then $g \in \mathcal{O}(\overline{D})$ and g peaks at (a,b). Consequently, $M_2 \subset \partial_B D \subset \partial_S D$.

- By (c), $M_3 \subset \partial_B D \subset \partial_S D$.

- For every $a = re^{i\varphi}$ with $r \in \{\frac{1}{2}, 1\}$, $0 < \varphi < \frac{\pi}{2}$ there exists a function $\psi \in \mathcal{O}(\overline{A})$ such that $\psi(a) = 1$ and $|\psi| < 1$ on $\overline{A} \setminus \{a\}$. Hence $\partial_B D \cap (\{a\} \times \overline{\mathbb{D}}) \neq \varnothing$. Now, by (*) with $\Omega := D$, $\Sigma := \{re^{i\varphi} : r \in \{\frac{1}{2}, 1\},\ 0 < \varphi < \frac{\pi}{2}\} \times \mathbb{D}$, we conclude that $M_4 \subset \partial_B D \subset \partial_S D$.

The remaining part of ∂D, i.e. the set $\Sigma := \partial D \setminus (M_0 \cup M_1 \cup M_2 \cup M_3 \cup M_4)$, is open in ∂D. It remains to use (*). □