ON THE BŁOCKI–ZWONEK CONJECTURES

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Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, $n \geq 1$. Furthermore, let K_Ω be the Bergman kernel for Ω, and let $g_\Omega(z,a)$ be the pluricomplex Green function with pole at $a \in \Omega$. Recall that

$$K_\Omega(z) = \sup \left\{ |f(z)|^2 : f \in \mathcal{O}(\Omega), \int_\Omega |f|^2 \, d\lambda_n \leq 1 \right\} ,$$

where λ_n is the Lebesgue measure in $\mathbb{C}^n$, and

$$g_\Omega(z,a) = \sup \left\{ u \in \mathcal{PSH}(\Omega)^- : u \leq \log | \cdot - a | + C \text{ near } a \right\} .$$

Błocki proved the following: *for $a \in \Omega$ and $t \leq 0$*

$$K_\Omega(a) \geq \frac{1}{e^{-2nt}\lambda_n(\{z \in \Omega : g_\Omega(z,a) < t\})} . \tag{0.1}$$

This lower bound is sharp in the sense that we have equality in the case when Ω is a ball centered at z. It is interesting to know whether the right-hand side of (0.1) is monotone in t. Therefore Błocki and Zwonek stated the following two conjectures.

The Błocki–Zwonek conjectures:

(1) *The function given by*

$$\alpha = \alpha_{\Omega,a} : (-\infty, 0) \ni t \mapsto \alpha(t) = e^{-2nt}\lambda_n(\{z \in \Omega : g_\Omega(z,a) < t\})$$

is nondecreasing.

(2) *The function defined by*

$$\beta = \beta_{\Omega,a} : (-\infty, 0) \ni t \mapsto \beta(t) = \log \left(\lambda_n(\{z \in \Omega : g_\Omega(z,a) < t\}) \right)$$

is convex.

Conjecture (2) implies (1) for hyperconvex domains. Conjecture (2) is not always true by an example due to Fornaess. An affirmative answer to conjecture (1) was given by Błocki and Zwonek for the case when $n = 1$.

Our main theorem is the following:

Theorem 1. *Let D be a bounded, balanced, pseudoconvex domain in $\mathbb{C}^n$, and let Ω be a pseudoconvex domain. If there exists a biholomorphic map $f : \Omega \to D$ such that $f(a) = 0$, $a \in \Omega$, then Conjecture (1), and (2), are true for Ω.*

Let us now recall the definitions of the classical Cartan domains:

$$\mathfrak{R}_I = \{A \in M(p,q) : I_p - AA^* > 0\} \subset \mathbb{C}^{pq};$$

$$\mathfrak{R}_{II} = \{A \in M(p,p) : A^T = A, I_p - AA^* > 0\} \subset \mathbb{C}^{\binom{p}{2}};$$

$$\mathfrak{R}_{III} = \{A \in M(p,p) : A^T = -A, I_p - AA^* > 0\} \subset \mathbb{C}^{\binom{p+1}{2}};$$

$$\mathfrak{R}_{IV} = \{z \in \mathbb{C}^n : \sqrt{\|z\|^2 + \sqrt{\|z\|^4 - |<z,\bar{z}>|^2}} < 1\}\,.$$

Here $M(p,q)$ denotes the set of all complex matrices with p rows and q columns, A^T denotes transposed matrix, $A^* = \overline{A^T}$ denotes the complex conjugate of the transposed matrix, and I_p denotes the identity matrix of order p. It should be noted that $\mathfrak{R}_{IV}$ is also known as the Lie ball, $\mathbb{L}_n$.

Corollary 2. *Conjecture (1), and Conjecture (2), are true for the above Cartan domains.*

Let $\mathfrak{D}_1$ be the family of domains for which conjecture (1) is true, and in the same manner let $\mathfrak{D}_2$ be the family of domains for which conjecture (2) holds.

Proposition 3. *For $j = 1, \dots, k$, let $\Omega_j \subset \mathbb{C}^{n_j}$ be bounded domains. Then we have the following properties.*

(1) *If $\Omega_j \in \mathfrak{D}_1$ for $j = 1, \dots, k$, then $\Omega_1 \times \cdots \times \Omega_k \in \mathfrak{D}_1$;*

(2) *If $\Omega_j \in \mathfrak{D}_2$ for $j = 1, \dots, k$, then $\Omega_1 \times \cdots \times \Omega_k \in \mathfrak{D}_2$.*

The Błocki–Zwonek conjectures are false for the Green function with two poles defined on the unit disc in $\mathbb{C}$.

References

[AC1] P. Åhag, R. Czyż, On the Błocki-Zwonek conjectures, Complex Var. Elliptic Equ. vol. 60 (2015), 1270-1276.

[AC2] P. Åhag, R. Czyż, On the Błocki-Zwonek conjectures and beyond, Arch. Math. vol. 105 (2015), 371-380.