PROPER HOLOMORPHIC MAPPINGS BETWEEN GENERALIZED HARTOGS TRIANGLES

PAWEŁ ZAPAŁOWSKI

1. Main results

The aim of this paper is to answer all questions posed by Jarnicki and Pflug in [6], Sections 2.5.2 and 2.5.3, concerning proper holomorphic mappings between generalized Hartogs triangles and holomorphic automorphisms of such domains.

Let $n, m \in \mathbb{N}$. For $p = (p_1, \dots, p_n) \in \mathbb{R}^n_{>0}$ and $q = (q_1, \dots, q_m) \in \mathbb{R}^m_{>0}$ define the *generalized Hartogs triangle* as

$$\mathbb{F}_{p,q} := \Big\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \sum_{j=1}^{n} |z_j|^{2p_j} < \sum_{j=1}^{m} |w_j|^{2q_j} < 1\Big\}.$$

If $n = m = 1$, then $\mathbb{F}_{1,1}$ is the standard Hartogs triangle.

Let $p, \tilde{p} \in \mathbb{R}^n_{>0}$, $q, \tilde{q} \in \mathbb{R}^m_{>0}$. The problem of characterization of proper holomorphic mappings

$$\mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}} \tag{1}$$

and the group $\mathrm{Aut}(\mathbb{F}_{p,q})$ of holomorphic automorphisms of $\mathbb{F}_{p,q}$ has been investigated in many papers (see, e.g., [8], [3], [4], [1], [2]).

Here is some notation. Let Σ_n denote the group of the permutations of the set $\{1, \dots, n\}$. For $\sigma \in \Sigma_n$, $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ denote $z_\sigma := (z_{\sigma(1)}, \dots, z_{\sigma(n)})$ and $\Sigma_n(z) := \{\sigma \in \Sigma_n : z_\sigma = z\}$. We shall also write $\sigma(z) := z_\sigma$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n_{>0}$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n_{>0}$ we shall write $\alpha\beta := (\alpha_1\beta_1, \dots, \alpha_n\beta_n)$ and $1/\beta := (1/\beta_1, \dots, 1/\beta_n)$. If, moreover, $\alpha \in \mathbb{N}^n$, then

$$\Psi_\alpha(z) := z^\alpha := (z_1^{\alpha_1}, \dots, z_n^{\alpha_n}), \quad z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

Theorem 1. *Let $n = m = 1$, $p, q, \tilde{p}, \tilde{q} \in \mathbb{R}_{>0}$.*

(a) There exists a proper holomorphic mapping $\mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ if and only if there exist $k, l \in \mathbb{N}$ such that

$$\frac{l\tilde{q}}{\tilde{p}} - \frac{kq}{p} \in \mathbb{Z}.$$

(b) A mapping $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ is proper and holomorphic if and only if

$$F(z,w) = \begin{cases} \left(\zeta z^k w^{l\tilde{q}/\tilde{p} - kq/p}, \xi w^l\right), & \text{if } q/p \notin \mathbb{Q} \\ \left(\zeta z^{k'} w^{l\tilde{q}/\tilde{p} - k'q/p} B\left(z^{p'} w^{-q'}\right), \xi w^l\right), & \text{if } q/p \in \mathbb{Q} \end{cases}, \quad (z,w) \in \mathbb{F}_{p,q},$$

where $\zeta, \xi \in \mathbb{T}$, $k, l \in \mathbb{N}$, $k' \in \mathbb{N} \cup \{0\}$ are such that $l\tilde{q}/\tilde{p} - kq/p \in \mathbb{Z}$, $l\tilde{q}/\tilde{p} - k'q/p \in \mathbb{Z}$, $p', q' \in \mathbb{N}$ are relatively prime with $p/q = p'/q'$, and B is a finite Blaschke product non-vanishing at 0 (if $B \equiv 1$, then $k' > 0$).

In particular, there are non-trivial proper holomorphic self-mappings in $\mathbb{F}_{p,q}$.

(c) $F \in \mathrm{Aut}(\mathbb{F}_{p,q})$ if and only if

$$F(z,w) = \left(w^{q/p}\phi\left(zw^{-q/p}\right), \xi w\right), \quad (z,w) \in \mathbb{F}_{p,q},$$

where $\xi \in \mathbb{T}$, and $\phi \in \mathrm{Aut}(\mathbb{D})$ (moreover, $\phi(0) = 0$ whenever $q/p \notin \mathbb{N}$).

Remark 2. The counterpart of the Theorem 1 for $p, q, \tilde{p}, \tilde{q} \in \mathbb{N}$ was proved (with minor mistakes) in [8], where it was claimed that a mapping $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ is proper and holomorphic if and only if

$$F(z,w) = \begin{cases} \left(\zeta z^k w^{l\tilde{q}/\tilde{p} - kq/p}, \xi w^l\right), & \text{if } q/p \notin \mathbb{N},\ l\tilde{q}/\tilde{p} - kq/p \in \mathbb{Z} \\ \left(\zeta w^{l\tilde{q}/\tilde{p}} B\left(zw^{-q/p}\right), \xi w^l\right), & \text{if } q/p \in \mathbb{N},\ l\tilde{q}/\tilde{p} \in \mathbb{N} \end{cases}, \tag{2}$$

where $\zeta, \xi \in \mathbb{T}$, $k, l \in \mathbb{N}$, and B is a finite Blaschke product. Nevertheless, the mapping

$$\mathbb{F}_{2,3} \ni (z,w) \longmapsto \left(z^3 w^3 B\left(z^2 w^{-3}\right), w^3\right) \in \mathbb{F}_{2,5},$$

where B is non-constant finite Blaschke product non-vanishing at 0, is proper holomorphic but not of the form (2). In fact, from the Theorem 1 (b) it follows immediately that for any choice of $p, q, \tilde{p}, \tilde{q} \in \mathbb{N}$ one may find a proper holomorphic mapping $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ having, as a factor of the first component, non-constant Blaschke product non-vanishing at 0.

Theorem 3. *Let $n = 1$, $m \geq 2$, $p, \tilde{p} \in \mathbb{R}_{>0}$, $q, \tilde{q} \in \mathbb{R}^m_{>0}$.*

(a) There exists a proper holomorphic mapping $\mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ if and only if there exists $\sigma \in \Sigma_m$ such that

$$\frac{p}{\tilde{p}} \in \mathbb{N} \quad \text{and} \quad \frac{q_\sigma}{\tilde{q}} \in \mathbb{N}^m.$$

(b) A mapping $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ is proper and holomorphic if and only if

$$F(z, w) = (\zeta z^k, h(w)), \quad (z, w) \in \mathbb{F}_{p,q},$$

where $\zeta \in \mathbb{T}$, $k \in \mathbb{N}$, and $h : \mathbb{E}_q \longrightarrow \mathbb{E}_{\tilde{q}}$ is proper and holomorphic such that $h(0) = 0$ (cf. Theorem 7).

In particular, there are non-trivial proper holomorphic self-mappings in $\mathbb{F}_{p,q}$.

(c) $F \in \mathrm{Aut}(\mathbb{F}_{p,q})$ if and only if

$$F(z, w) = (\zeta z, h(w)), \quad (z, w) \in \mathbb{F}_{p,q},$$

where $\zeta \in \mathbb{T}$, $h \in \mathrm{Aut}(\mathbb{E}_q)$, $h(0) = 0$ (cf. Theorem 7).

Theorem 4. *Let $n \geq 2$, $m = 1$, $p = (p_1, \dots, p_n), \tilde{p} = (\tilde{p}_1, \dots, \tilde{p}_n) \in \mathbb{R}^n_{>0}$, $q, \tilde{q} \in \mathbb{R}_{>0}$.*

(a) There exists a proper holomorphic mapping $\mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ if and only if there exist $\sigma \in \Sigma_n$ and $r \in \mathbb{N}$ such that

$$\frac{p_\sigma}{\tilde{p}} \in \mathbb{N}^n \quad \text{and} \quad \frac{r\tilde{q} - q}{\tilde{p}_j} \in \mathbb{Z}, \quad j = 1, \dots, n.$$

(b) A mapping $F = (G_1, \dots, G_n, H) : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ is proper and holomorphic if and only if

$$\begin{cases} G_j(z, w) = w^{r\tilde{q}/\tilde{p}_j} f_j\left(z_1 w^{-q/p_1}, \dots, z_n w^{-q/p_n}\right), & j = 1, \dots, n, \\ H(z, w) = \xi w^r, \end{cases} \quad (z, w) \in \mathbb{F}_{p,q},$$

where $(f_1, \dots, f_n) : \mathbb{E}_p \longrightarrow \mathbb{E}_{\tilde{p}}$ is proper and holomorphic (cf. Theorem 7), $\xi \in \mathbb{T}$, and $r \in \mathbb{N}$ is such that $(r\tilde{q} - q)/\tilde{p}_j \in \mathbb{Z}$, $j = 1, \dots, n$. Moreover, if there is a j such that $1/\tilde{p}_j \in \mathbb{N}$, then $q \in \mathbb{N}$ and $r\tilde{q}/\tilde{p}_j \in \mathbb{N}$ whenever $1/\tilde{p}_j \in \mathbb{N}$.

In particular, there are non-trivial proper holomorphic self-mappings in $\mathbb{F}_{p,q}$.

(c) $F = (G_1, \dots, G_n, H) \in \mathrm{Aut}(\mathbb{F}_{p,q})$ if and only if

$$\begin{cases} G_j(z, w) = w^{q/p_j} g_j\left(z_1 w^{-q/p_1}, \dots, z_n w^{-q/p_n}\right), & j = 1, \dots, n, \\ H(z, w) = \xi w, \end{cases} \quad (z, w) \in \mathbb{F}_{p,q},$$

where $(g_1, \dots, g_n) \in \mathrm{Aut}(\mathbb{E}_p)$ (cf. Theorem 7), $\xi \in \mathbb{T}$.

Remark 5. It should be mentioned, that although the structure of the automorphism group $\mathrm{Aut}(\mathbb{F}_{p,q})$ does not change when passing from $p \in \mathbb{N}^n$, $q \in \mathbb{N}$ to $p \in \mathbb{R}^n_{>0}$, $q > 0$, the class of proper holomorphic mappings $\mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ does. It is a consequence of the fact that the structure of the proper holomorphic mappings $\mathbb{E}_p \longrightarrow \mathbb{E}_{\tilde{p}}$ changes when passing from $p, \tilde{p} \in \mathbb{N}^n$ to $p, \tilde{p} \in \mathbb{R}^n_{>0}$ (see Section 2).

Theorem 6. *Let $n, m \geq 2$, $p, \tilde{p} \in \mathbb{R}^n_{>0}$, $q, \tilde{q} \in \mathbb{R}^m_{>0}$.*

(a) There exists a proper holomorphic mapping $\mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ if and only if there exist $\sigma \in \Sigma_n$ and $\tau \in \Sigma_m$ such that

$$\frac{p_\sigma}{\tilde{p}} \in \mathbb{N}^n \quad \text{and} \quad \frac{q_\tau}{\tilde{q}} \in \mathbb{N}^m.$$

(b) A mapping $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ is proper and holomorphic if and only if

$$F(z, w) = (g(z), h(w)), \quad (z, w) \in \mathbb{F}_{p,q},$$

where mappings $g : \mathbb{E}_p \longrightarrow \mathbb{E}_{\tilde{p}}$ and $h : \mathbb{E}_q \longrightarrow \mathbb{E}_{\tilde{q}}$ are proper and holomorphic such that $g(0) = 0$, $h(0) = 0$ (cf. Theorem 7).

In particular, every proper holomorphic self-mapping in $\mathbb{F}_{p,q}$ is an automorphism.

(c) $F \in \mathrm{Aut}(\mathbb{F}_{p,q})$ if and only if

$$F(z, w) = (g(z), h(w)), \quad (z, w) \in \mathbb{F}_{p,q},$$

where $g \in \mathrm{Aut}(\mathbb{E}_p)$, $h \in \mathrm{Aut}(\mathbb{E}_q)$ with $g(0) = 0$, $h(0) = 0$ (cf. Theorem 7).

2. Complex ellipsoids

For $p = (p_1, \dots, p_n) \in \mathbb{R}^n_{>0}$, define the *complex ellipsoid*

$$\mathbb{E}_p := \Big\{(z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^{2p_j} < 1\Big\}.$$

Note that $\mathbb{E}_{(1,\dots,1)}$ is the unit ball in $\mathbb{C}^n$. Moreover, if $p/q \in \mathbb{N}^n$, then $\Psi_{p/q} : \mathbb{E}_p \longrightarrow \mathbb{E}_q$ is proper and holomorphic.

The problem of characterization of proper holomorphic mappings between two given complex ellipsoids has been investigated in [7] and [5].

Theorem 7. *Assume that $n \geq 2$, $p, q \in \mathbb{R}^n_{>0}$.*

(a) (cf. [7], [5]). There exists a proper holomorphic mapping $\mathbb{E}_p \longrightarrow \mathbb{E}_q$ if and only if there exists $\sigma \in \Sigma_n$ such that

$$\frac{p_\sigma}{q} \in \mathbb{N}^n.$$

(b) A mapping $F : \mathbb{E}_p \longrightarrow \mathbb{E}_q$ is proper and holomorphic if and only if

$$F = \Psi_{p_\sigma/(qr)} \circ \phi \circ \Psi_r \circ \sigma,$$

where $\sigma \in \Sigma_n$ is such that $p_\sigma/q \in \mathbb{N}^n$, $r \in \mathbb{N}^n$ is such that $p_\sigma/(qr) \in \mathbb{N}^n$, and $\phi \in \operatorname{Aut}(\mathbb{E}_{p_\sigma/r})$.

In particular, every proper holomorphic self-mapping in $\mathbb{E}_p$ is an automorphism.

(c) (cf. [7], [5]). If $0 \leq k \leq n$, $p \in \{1\}^k \times (\mathbb{R}_{>0} \setminus \{1\})^{n-k}$, $z = (z', z_{k+1}, \dots, z_n)$, then $F = (F_1, \dots, F_n) \in \operatorname{Aut}(\mathbb{E}_p)$ if and only if

$$F_j(z) = \begin{cases} H_j(z'), & \text{if } j \leq k \\ \zeta_j z_{\sigma(j)} \left(\frac{\sqrt{1-\|a'\|^2}}{1-\langle z', a'\rangle}\right)^{1/p_{\sigma(j)}}, & \text{if } j > k \end{cases},$$

where $\zeta_j \in \mathbb{T}$, $j > k$, $H = (H_1, \dots, H_k) \in \operatorname{Aut}(\mathbb{B}_k)$, $a' = H^{-1}(0)$, and $\sigma \in \Sigma_n(p)$.

3. Boundary behavior of proper holomorphic mappings between Hartogs triangles

Note that the boundary $\partial\mathbb{F}_{p,q}$ of the generalized Hartogs triangle $\mathbb{F}_{p,q}$ may be written as $\partial\mathbb{F}_{p,q} = \{0, 0\} \cup K_{p,q} \cup L_{p,q}$, where

$$K_{p,q} := \Big\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : 0 < \sum_{j=1}^n |z_j|^{2p_j} = \sum_{j=1}^m |w_j|^{2q_j} < 1\Big\},$$

$$L_{p,q} := \Big\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \sum_{j=1}^n |z_j|^{2p_j} < \sum_{j=1}^m |w_j|^{2q_j} = 1\Big\}.$$

Let $\mathbb{F}_{p,q}$ and $\mathbb{F}_{\tilde{p},\tilde{q}}$ be two generalized Hartogs triangles and let $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ be proper holomorphic mapping. It is known ([8], [3]) that F extends holomorphically through any boundary point $(z_0, w_0) \in \partial\mathbb{F}_{p,q} \setminus \{(0, 0)\}$.

The aim of this section is to prove the following crucial fact.

Lemma 8. *Let $nm \neq 1$. If $F : \mathbb{F}_{p,q} \longrightarrow \mathbb{F}_{\tilde{p},\tilde{q}}$ is proper and holomorphic, then*

$$F(K_{p,q}) \subset K_{\tilde{p},\tilde{q}}, \quad F(L_{p,q}) \subset L_{\tilde{p},\tilde{q}}.$$

The following two lemmas will be needed in the proof of Lemma 8.

Lemma 9. *If $n \geq 2$ and $m = 1$, then $K_{p,q}$ is not Levi flat at $(z, w) \in K_{p,q}$, where at lest two coordinates of z are non-zero (i.e. the Levi form of the defining function restricted to the complex tangent space is not degenerate at (z, w)).*

Lemma 10. *Let $D \subset \mathbb{C}^{n+1}$ and $V \subset \mathbb{C}^n$ be bounded domains, $a \in V$, and let $\Phi : V \longrightarrow \partial D$ be holomorphic mapping such that $\operatorname{rank} \Phi'(a) = n$. Assume that D has local defining function r of class $\mathcal{C}^2$ in the neighborhood of $\Phi(a)$. Then ∂D is Levi flat at $\Phi(a)$.*

References

1. Z. H. Chen, *Proper holomorphic mappings between some generalized hartogs triangles*, Geometric function theory in several complex variables. Proceedings of a satellite conference to the international congress of mathematicians, ICM-2002, Beijing, China, August 30–September 2, 2002 (C. H. Fitzgerald and S. Gong, eds.), River Edge, NJ: World Scientific, 2004, pp. 74–81.
2. Z. H. Chen and Y. Liu, *A note on the classification of the proper mappings between some generalized Hartogs triangles*, Chin. J. Contemp. Math. **24** (2003), no. 3, 215–220.
3. Z. H. Chen and D. K. Xu, *Proper holomorphic mappings between some nonsmooth domains*, Chin. Ann. Math. Ser. B **22** (2001), no. 2, 177–182.
4. ———, *Rigidity of proper self-mapping on some kinds of generalized Hartogs triangle*, Acta Math. Sin. (Engl. Ser.) **18** (2002), no. 2, 357–362.
5. G. Dini and A. Selvaggi Primicerio, *Proper holomorphic mappings between generalized pseudoellipsoids*, Ann. Mat. Pura Appl. (4) **158** (1991), 219–229.
6. M. Jarnicki and P. Pflug, *First steps in several complex variables: Reinhardt domains*, EMS Textbooks in Mathematics, European Mathematical Society Publishing House, 2008.
7. M. Landucci, *On the proper holomorphic equivalence for a class of pseudoconvex domains*, Trans. Amer. Math. Soc. **282** (1984), 807–811.
8. ———, *Proper holomorphic mappings between some nonsmooth domains*, Ann. Mat. Pura Appl. (4) **155** (1989), 193–203.