KELLOG-WARSCHAWSKI THEOREM

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During seminar I presented proof of theorem, that gives us some information about smoothness of conformal map in relation to smoothnes of its image's border.

Theorem 0.1 (Kellog-Warschawski Theorem [1]). *Let f map $\mathbb{D}$ conformally onto the inner domain of the Jordan curve C of class $\mathcal{C}^{n,\alpha}$ where $n = 1, 2, \dots$ and $0 < \alpha < 1$. Then $f^{(n)}$ has a continuous extention to $\overline{\mathbb{D}}$ and for some constant M*

$$|f^{(n)}(z_1) - f^{(n)}(z_2)| \leq M|z_1 - z_2|^{\alpha} \qquad \textit{for} \quad z_1, z_2 \in \overline{\mathbb{D}}.$$

Proof was based on [2], main step was to prove following theorem.

Theorem 0.2 (Theorem 3.5 from [2]). *Let f map $\mathbb{D}$ conformally onto the inner domain of Dini-smooth Jordan curve C. Then f' has a continuous extention to $\overline{\mathbb{D}}$ and for some constant M*

$$\frac{f(\zeta) - f(z)}{\zeta - z} \to f'(z) \neq 0 \qquad \textit{for} \quad \zeta \to z, \quad \zeta, z \in \overline{\mathbb{D}},$$

$$|f^{(n)}(z_1) - f^{(n)}(z_2)| \leq M\omega^*(\delta) \qquad \textit{for} \quad z_1, z_2 \in \overline{\mathbb{D}}, \quad |z_1 - z_2| \leq \delta.$$

Where ω^* is defined as following: let $w(\tau)$ be parametrization of curve C, $w'(\tau)$ is Dini-continuous because C is Dini-smooth. Let $\omega(\delta)$ denote modulus of continuity of w' then

$$\omega^*(\delta) \equiv \int_0^{\delta} \frac{w(t)}{t} dt + \delta \int_{\delta}^{\pi} \frac{w(t)}{t^2} dt.$$

References

[1] S.E. Warschawski *ber das Randverhalten der Ableitung der Abbildungsfunktion bei konformer Abbildung.* Mathematische Zeitschrift 35 (1932) 321-456

[2] C.H. Pommerenke *Boundary behaviour of conformal maps* Grundlehren der mathematischen Wissenschaften 299 (1992)