Tangents and corners

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G - any simply connected domain in $\mathbb{C}$ with locally connected boundary
$\mathbb{D} = \{z \in \mathbb{C} :| z |< 1\} = D(0,1)$ unit disc
$\mathbb{T} = \{z \in \mathbb{C} :| z |= 1\} = \partial D$ unit circle

Definition: We say that ∂G has a corner of opening $\alpha\pi$ at $f(\gamma) \neq \infty$ $(0 \leqslant \alpha \leqslant 2\pi)$ if

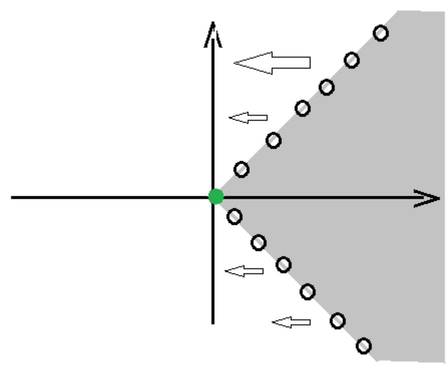
$$arg[f(e^{it}) - f(e^{i\theta})] \rightarrow \begin{cases} \beta & t \rightarrow \theta^+ \\ \beta + \alpha\pi & t \rightarrow \theta^- \end{cases}$$

Example: Let f be conformal map of $\mathbb{D}$ onto the sector $G = \{| arg\omega |< \frac{\pi\alpha}{2}\}$

$f_{(z)} = \left(\frac{1-z}{1+z}\right)^\alpha = e^{-i\pi\alpha}\left(\frac{z-1}{z+1}\right)^\alpha$ for $(0 < \alpha \leqslant 2)$

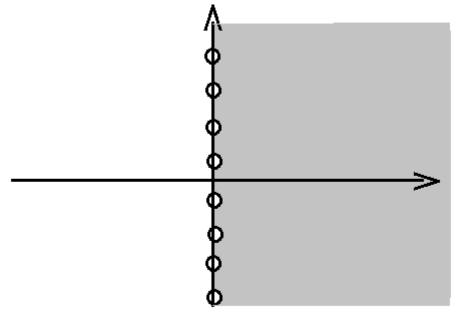
There are four posibilities:

1. $\alpha \in (0,1)$



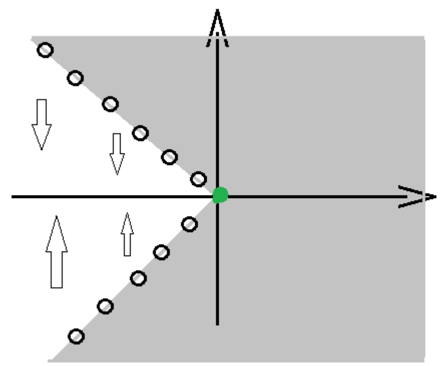
   There is an outward-pointing cusp at 0

2. $\alpha = 1$



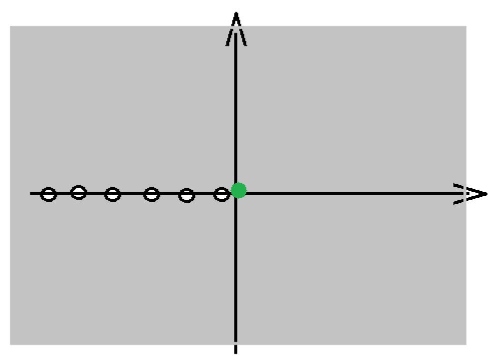
There is a tangent of direction angle $\frac{\pi}{2}$

3. $\alpha \in (1, 2)$



There is an inward-pointing cusp at 0

4. $\alpha = 2$



There is an inward-pointing cusp at 0

Theorem: Let f maps $\mathbb{D}$ conformally onto G where ∂G is locally connected. Let $\gamma = e^{i\theta} \in \mathbb{T}$ and $f(\gamma) \neq \infty$. Then ∂G has a corrner of opening $\pi\alpha(0 \leqslant \alpha \leqslant 2)$ at $f(\gamma)$ if and only if

$$arg\frac{f(z)-f(\gamma)}{(z-\gamma)^{\alpha}} \to \beta - \alpha(\theta + \frac{\pi}{2}) \text{ as } z \to \gamma,\ z \in \mathbb{D}.$$

REFERENCES

[1]Christian Pommerenke, Fachbereich, Technische Universitat, 1000 Berlin 12, FRG