Countability via capacity

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The aim of my presentation is to use logarithmic capacity to characterize thoes compact sets that are countable.

Definition 1. *Polar set*
Let E be a subset of $\mathbb{C}$, if there exists a subharmonic function u, which domain D containins E, such that:

$$\begin{cases} u \equiv -\infty & on \quad E \\ u \not\equiv -\infty & on \quad D \end{cases}$$

then the set E is called polar.

Definition 2. *Logarithmic capacity*
Let K be a compact subset of $\mathbb{C}$. The logarithmic capacity of K is definied by

$$c(K) := exp\big(\sup_{\mu} \iint log|z-w| d\mu(z) d\mu(w)\big),$$

where the supremum is taken over all Borel probability measures μ, that are supported on K.

It is well known, that K is polar if and only if $c(K) = 0$.

Theorem 1. *Let K be a compact subset of $\mathbb{C}$. Then following are equivalent:*

i. $c(K + L) = 0$ whenever $c(L) = 0$;

ii. K is countable.

Proof. $(ii) \Rightarrow (i)$
If K is a countable set then $K + L$ is a countable union of translates of L.
If E is a compact subset of $\mathbb{R}$ then for all $w \in \mathbb{C}$ we have that $c(E + w) = c(E)$.
We can write then:

$$c(K + L) = c(\bigcup_{n=1} \{k_n\} + L) = c(L) = 0$$

$$c(K + L) = 0$$

□

The main idea of the proof of the relation $(i) \Rightarrow (ii)$ in Theorem 1 is to reduce it to certain types of Cantor sets (the generalized Cantor sets, regular linear Cantor sets and binary linear Cantor sets), for which it is easy to make some explicit calculations. Below there are some properties of thoes sets.

Lemma 1. *Let B be a binary linear Cantor set. Then there exists a regular linear Cantor set C such that $c(C) = 0$ and $c(B \times C) > 0$.*

Lemma 2. *Let X, Y be a compact subsets of $\mathbb{R}$ such that $c(X \times Y) > 0$. Then there exist $\lambda > 0$ such that $c(X + \lambda Y) > 0$.*

Lemma 3. *Let Z ba a uncountable compact subset of $\mathbb{R}$. Then there exists a Lipschitz homeomorphism $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(Z)$ contains a binary linear Cantor set.*

One of the applications of Theorem 1 is using it to prove the so-called scarcity theorem for countable analytic multifunctions. The proof becomes much shorter than the original one.

References

1. N. Levenberg, T. J. Ransford, J. Rostand, Z. Słodkowski, *Countability via capacity*, Mathematische Zeitschrift 242 (2002), 399-406.