The Koebe Distortion Theorem

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The class S consists of all functions: $f(z) = z + a_2 z^2 + a_3 z^3 + \dots \quad (|z| < 1)$ analytic and univalent in $\mathbb{D}$. Many formulas take their nicest form with this normalization $f(0) = 0, f'(0) = 1$. The class Σ consists of all functions: $g(\zeta) = \zeta + b_0 + b_1 \zeta^{-1} + \dots \quad (|\zeta| > 1)$ univalent in $\mathbb{D}^*$.

We can show that:

$$\text{area}(\mathbb{C} \backslash g(\mathbb{D}^*)) = \pi \left(1 - \sum_{n=1}^{\infty} n |b_n|^2 \right) \quad \text{for} \quad g \in \Sigma \, .$$

This implies the area theorem:

Theorem 1 (Area Theorem). If $g \in \Sigma$, then: $\sum_{n=1}^{\infty} n|b_n|^2 \leq 1$, with equality if and only if $g \in \widetilde{\Sigma}$.

Proposition 1. If f maps $\mathbb{D}$ conformally into $\mathbb{C}$ then

$$\left| (1 - |z|^2) \frac{f''(z)}{f'(z)} - 2\overline{z} \right| \leq 4 \quad \text{for} \quad z \in \mathbb{D}.$$

Equality holds for the Koebe function. Integrating this inequality twice we obtain the famous Koebe distortion theorem.

Theorem 2 (Distortion Theorem). For any $f \in S$, if $|z| = q$,

$$\frac{1-q}{(1+q)^3} \leq |f'(z)| \leq \frac{1+q}{(1-q)^3} \, , \quad 0 \leq q < 1 \, .$$

Theorem 3 (Growth Theorem). For any $f \in S$, if $|z| = q$,

$$\frac{q}{(1+q)^2} \leq |f(z)| \leq \frac{q}{(1-q)^2} \, , \quad 0 \leq q < 1 \, .$$

Theorem 4 (Koebe Distortion Theorem). If f maps $\mathbb{D}$ conformally into $\mathbb{C}$ and if $z \in \mathbb{D}$ then

$$|f'(0)| \frac{|z|}{(1+|z|)^2} \leq |f(z) - f(0)| \leq |f'(0)| \frac{|z|}{(1-|z|)^2} \, ,$$

$$|f'(0)| \frac{1-|z|}{(1+|z|)^3} \leq |f'(z)| \leq |f'(0)| \frac{1+|z|}{(1-|z|)^3} \, .$$

References

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