FAMILIES OF STRICTLY PSEUDOCONVEX DOMAINS AND PEAK FUNCTIONS

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Let $D \subset \mathbb{C}^n$ be a bounded domain and let ζ be a boundary point of D. It is called a *peak point* with respect to $\mathcal{O}(\overline{D})$, the family of functions which are holomorphic in a neighborhood of $\overline{D}$, if there exist a function $f \in \mathcal{O}(\overline{D})$ such that $f(\zeta) = 1$ and $f(\overline{D} \setminus \{\zeta\}) \subset \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Such a function is a *peak function for D at ζ*. The concept of peak functions appears to be a powerful tool in complex analysis with many applications. It has been used to show the existence of (complete) proper holomorphic embeddings of strictly pseudoconvex domains into the unit ball $\mathbb{B}^N$ with large N (see [5],[3]), to estimate the boundary behavior of Carathéodory and Kobayashi metrics ([1],[6]), or to construct the solution operators for $\overline{\partial}$ problem with L^∞ or Hölder estimates ([4],[10]), just to name a few of those applications.

It is well known that every boundary point of strictly pseudoconvex domain is a peak point. Even more is true, in [6] it is showed that, given a strictly pseudoconvex domain G, there exists an open neighborhood $\widehat{G}$ of G, and a continuous function $h : \widehat{G} \times \partial G \to \mathbb{C}$ such that for $\zeta \in \partial G$, the function $h(\cdot;\zeta)$ is a peak function for G at ζ.

In a recent paper [2] the following question has been posed:

Problem 0.1. Let $\rho : \mathbb{D} \times \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic function of class $\mathcal{C}^{2+k}, k \in \mathbb{N} \cup \{0\}$, such that for any $z \in \mathbb{D}$ the truncated function $\rho|_{\{z\}\times\mathbb{C}^n}$ is strictly plurisubharmonic. Define $G_z := \{w \in \mathbb{C}^n : \rho(z, w) < 0\}, z \in \mathbb{D}$. This can be understood as a family of strictly pseudoconvex domains over $\mathbb{D}$. Does there exist a $\mathcal{C}^k$-continuously varying family $(h_{z,\zeta})_{z\in\mathbb{D},\zeta\in\partial G_z}$ of peak functions for G_z at ζ?

We answer this question affirmatively in the case $k = 0$ and under additional assumption that, roughly speaking, the function ρ keeps its regularity up to the set $\Omega \times \mathbb{C}^n$, where Ω is some open neighborhood of $\overline{\mathbb{D}}$. Namely, let us consider the following:

Situation 0.2. Let $(G_t)_{t\in T}$ be a family of bounded strictly pseudoconvex domains, where T is a compact metric space with associated metric d. Suppose we have a domain $U \subset\subset \mathbb{C}^n$ such that

1. $\bigcup_{t\in T} \partial G_t \subset\subset U$,
2. for each $t \in T$ there exists on U a defining function r_t for G_t of class $\mathcal{C}^2$ and such that its Levi form $\mathcal{L}_{r_t}(\zeta; X)$ is strictly positive for any $\zeta \in \partial G_t$ and $X \in \mathbb{C}^n \setminus \{0\}$,

(3) for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $s, t \in T$ with $d(s,t) \leq \delta$ there is $\|r_t - r_s\|_{\mathcal{C}^2(U)} < \varepsilon$.

We shall prove the following:

Theorem 0.3. *Let $(G_t)_{t \in T}$ be a family of strictly pseudoconvex domains as in Situation 0.2. Then there exists an $\varepsilon > 0$ such that for any $\eta_1 < \varepsilon$ there exist an $\eta_2 > 0$ and positive constants d_1, d_2 such that for any $t \in T$ there exist a domain $\widehat{G_t}$ containing $\overline{G_t}$, and functions $h_t(\cdot;\zeta) \in \mathcal{O}(\widehat{G_t}), \zeta \in \partial G_t$ fulfilling the following conditions:*

(a) *$h_t(\zeta;\zeta) = 1, |h_t(\cdot;\zeta)| < 1$ on $\overline{G_t} \setminus \{\zeta\}$ (in particular, $h_t(\cdot;\zeta)$ is a peak function for G_t at ζ),*

(b) *$|1 - h_t(z;\zeta)| \leq d_1\|z - \zeta\|, z \in \widehat{G_t} \cap \mathbb{B}(\zeta, \eta_2)$,*

(c) *$|h_t(z;\zeta)| \leq d_2 < 1, z \in \overline{G_t}, \|z - \zeta\| \geq \eta_1$.*

Moreover, the constants $\varepsilon, \eta_2, d_1, d_2$, domains $\widehat{G_t}$, and functions $h_t(\cdot;\zeta)$ may be chosen in such a way that for any $\alpha > 0$ and any fixed triple (t_0, ζ_0, z_0), where $t_0 \in T, \zeta_0 \in \partial G_{t_0}$, and $z_0 \in \widehat{G_{t_0}}$, there exists a $\delta > 0$ such that whenever the triple (s, ξ, w) satisfies $s \in T, \xi \in \partial G_s, w \in \widehat{G_s}$, and $\max\{d(s,t_0), \|\xi - \zeta_0\|, \|w - z_0\|\} < \delta$, then $|h_{t_0}(z_0;\zeta_0) - h_s(w;\xi)| < \alpha$.

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