THE GEOMETRY OF m-HYPERCONVEX DOMAINS

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Let $\mathcal{SH}_m(\Omega)$ be the class of m-subharmonic functions on $\Omega \subset \mathbb{C}^n$, $1 \leq m \leq n$.

Definition 0.1. Let Ω be a domain in $\mathbb{C}^n$. We say that Ω is *m-hyperconvex* if it admits an exhaustion function that is negative and m-subharmonic.

Definition 0.2. Let Ω be a bounded domain in $\mathbb{C}^n$, and let μ be a non-negative regular Borel measure defined on $\bar{\Omega}$. We say that μ is a *Jensen measure with barycenter $z_0 \in \bar{\Omega}$ w.r.t. $\mathcal{SH}_m(\Omega) \cap \mathcal{C}(\bar{\Omega})$* if

$$u(z_0) \leq \int_{\bar{\Omega}} u \, d\mu \qquad \text{for all } u \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}(\bar{\Omega})\,.$$

The set of such measures will be denoted by $\mathcal{J}_{z_0}^m$. Furthermore, the *Jensen boundary* w.r.t. $\mathcal{J}_{z_0}^m$ is defined as

$$\partial_{\mathcal{J}^m} = \left\{ z \in \bar{\Omega} : \mathcal{J}_z^m = \{\delta_z\} \right\} .$$

The Jensen boundary is another name for the Choquet boundary w.r.t. a given class of Jensen measures.

We prove the following theorems.

Theorem 0.3. *Assume that Ω, Ω_1, and Ω_2 are bounded m-hyperconvex domains in $\mathbb{C}^n$, $n \geq 2$, $1 \leq m \leq n$. Then we have the following.*

(1) *If $\Omega_1 \cap \Omega_2$ is a domain then it is m-hyperconvex in $\mathbb{C}^n$.*

(2) *The domain $\Omega_1 \times \Omega_2$ is m-hyperconvex in $\mathbb{C}^{2n}$.*

(3) *The domain Ω admits a negative exhaustion function that is strictly m-subharmonic on Ω, and continuous on $\bar{\Omega}$.*

(4) *If Ω is a priori only a bounded domain in $\mathbb{C}^n$ such that for every $z \in \partial\Omega$ there exists a neighborhood U_z such that $\Omega \cap U_z$ is m-hyperconvex, then Ω is m-hyperconvex.*

Theorem 0.4. *Assume that Ω is a bounded domain in $\mathbb{C}^n$, $n \geq 2$, $1 \leq m \leq n$. Then the following assertions are equivalent.*

(1) *Ω is m-hyperconvex in the sense of Definition 0.1;*

(2) *$\partial\Omega$ has a weak barrier at every point that is m-subharmonic;*

(3) *Ω admits an exhaustion function that is negative, smooth and strictly m-subharmonic;*

(4) *for every $z \in \partial\Omega$, and every $\mu \in \mathcal{J}_z^m$, we have that* $\operatorname{supp}(\mu) \subseteq \partial\Omega$.

Theorem 0.5. *Assume that Ω is a bounded domain in $\mathbb{C}^n$, $n \geq 2$, $1 \leq m \leq n$. Then the following assertions are equivalent.*

(1) *$\partial\Omega$ is B_m-regular at every boundary point $z_0 \in \partial\Omega$, in the sense that*

$$\lim_{\substack{z\to z_0\\ z\in\Omega}} \mathrm{PB}_f^m(z) = f(z_0)\,,$$

for each continuous function $f : \partial\Omega \to \mathbb{R}$. Here

$$\mathrm{PB}_f^m(z) = \sup\left\{v(z) : v \in \mathcal{SH}_m(\Omega),\ \varlimsup_{\substack{\zeta\to\xi\\ \zeta\in\Omega}} v(\zeta) \leq f(\xi)\,,\ \ \forall\xi \in \partial\Omega\right\}.$$

(2) *$\partial\Omega$ has a strong barrier at every point that is m-subharmonic;*

(3) *Ω admits an exhaustion function φ that is negative, smooth, m-subharmonic, and such that*

$$\big(\varphi(z) - |z|^2\big) \in \mathcal{SH}_m(\Omega)\,;$$

(4) *$\partial\Omega = \partial_{\mathcal{J}_z^m}$ in the sense of Definition 0.2.*

References

[ACH] P. Åhag, R. Czyż, L. Hed, The geometry of m-hyperconvex domains, arXiv:1703.02796, 2017.