"Bishop's multivalued projections" - abstract

1. Research Project Objectives

Let M be a complex submanifold of a Stein manifold X. It is known (cf. e.g. [4], Chapter VIII, Section A, Theorem 18) that $\mathcal{O}(X)|_M = \mathcal{O}(M)$, i.e. each function holomorphic on M extends holomorphically to X. In [2] E. Bishop proposed a proof of the above result based on the use of special analytic polyhedra (without sheaves methods). The central part of Bishop's proof is to show that for every relatively compact domain $U \subset X$ with $U \cap M \neq \varnothing$ we have $\mathcal{O}(U)|_{U\cap M} = \mathcal{O}(M)|_{U\cap M}$. At the end of his proof E. Bishop suggested that an alternative proof may be performed in the language of holomorphic multivalued projections. The aim of the project is to continue the realization of this idea (the partial results are proved in [3]).

2. Significance of the project and work plan

Let X be a topological Hausdorff space. We define an equivalence relation on X^n by

$$(x_1, \dots, x_n) \sim (y_1, \dots, y_n) :\Longleftrightarrow (y_1, \dots, y_n) \text{ is a reordering of } (x_1, \dots, x_n).$$

Then we define the n-th symmetric product (cf. [3],[6]) of X: $\overleftrightarrow{X^n} := X^n/\sim$. In the case $n = 1$, we get $\overleftrightarrow{X^1} = X$.

Definition 1 (cf. [6])**.** Let Y be Hausdorff topological space and let $F : X \longrightarrow \overleftrightarrow{Y^n}$ be continuous. Then we put

$$X_F^{(k)} := \{x \in X : \#\{F(x)\} = k\},$$

$$\chi_F := \max\{k : X_F^{(k)} \neq \varnothing\}, \quad X_F := X_F^{(\chi_F)}.$$

Note that X_F is open.

Proposition 1 (cf. [6])**.** *Let F be as above. Suppose that*

$$a \in X_F, \quad F(a) = [b_1 : \mu_1, \dots, b_k : \mu_k], \quad k := \chi_F.$$

Then there is a neighborhood $U \subset X_F$ of a and there are uniquely defined continuous functions $f_i : U \longrightarrow Y$, $i = 1, \dots, k$, such that

$$F(x) = [f_1(x) : \mu_1, \dots, f_k(x) : \mu_k], \quad x \in U.$$

In the above situation, we will write $F = \mu_1 f_1 \oplus \cdots \oplus \mu_k f_k$ on U.

Definition 2 (cf. [3])**.** Let M, N be complex manifolds and let M be connected. We say a continuous function $F : M \longrightarrow \overleftrightarrow{N^n}$ is *holomorphic on M* ($F \in \mathcal{O}(M, \overleftrightarrow{N^n})$), if:

- $M \setminus M_F$ is thin, i.e. every point $x_0 \in M \setminus M_F$ has open connected neighborhood $V \subset M$ and a function $\varphi \in \mathcal{O}(V)$, $\varphi \not\equiv 0$, such that $(M \setminus M_F) \cap V \subset \varphi^{-1}(0)$,
- for every $a \in M_F$, if $F = \mu_1 f_1 \oplus \cdots \oplus \mu_k f_k$ on V as in Proposition 1, then $f_1, \dots, f_k \in \mathcal{O}(V)$.

If M is disconnected, then we say that F is *holomorphic on M*, if $F|_C \in \mathcal{O}(C, \overleftrightarrow{N^n})$ for any connected component $C \subset M$.

Definition 3 (cf. [3])**.** We say a holomorphic function

$$\Delta : U \longrightarrow \overleftrightarrow{(M \times \mathbb{C})^n}$$

is a *holomorphic multivalued projection* $U \longrightarrow M$, if for any $x \in U \cap M$ such that $\Delta(x) = \llbracket (x_1, z_1), \dots, (x_n, z_n) \rrbracket$ we have $x_{j_0} = x$ for some $j_0 \in \{1, \dots, n\}$ and $z_j = 0$ for any $j \in \{1, 2, \dots, n\} \setminus \{j_0\}$.

Let $\mathcal{P}$ denote set of all holomorphic multivalued projections $U \longrightarrow M$. Then we define the map $\Xi = \Xi_U : (U \cap M) \times \mathcal{P} \longrightarrow \mathbb{C}, \quad \Xi(x, \Delta) := z_{j_0}$.

We say $\Pi = (\Delta_j)_{j=1}^k$ is a *system of holomorphic multivalued projections* $U \longrightarrow M$, if $\Delta_j : U \longrightarrow \overleftrightarrow{(M \times \mathbb{C})^{n_j}}$, $j = 1, \dots, k$, are holomorphic multivalued projections and $\sum_{j=1}^k \Xi(x, \Delta_j) = 1$ for any $x \in U \cap M$.

In particular, we get an extension operator $L_\Pi : \mathcal{O}(M) \longrightarrow \mathcal{O}(U)$,

$$L_\Pi(u) := \sum_{s=1}^k \overleftrightarrow{u_s} \circ \Delta_s, \quad \overleftrightarrow{u_s}(\llbracket (x_1, \lambda_1), \dots, (x_{n_s}, \lambda_{n_s}) \rrbracket) := \sum_{j=1}^{n_s} u(x_j)\lambda_j.$$

The main result of [3] is the proof of the following "local" extension theorem.

Theorem 1 (cf. [3])**.** *There exists a system* Π *of multivalued holomorphic projections* $U \longrightarrow M$*. Consequently, there exists a linear continuous extension operator* $L_\Pi : \mathcal{O}(M) \longrightarrow \mathcal{O}(U)$*.*

A "global" version of Theorem 1 will be discussed in the project.

Definition 4. A sequence $\Pi = (\Delta_{s,j})_{(s,j) \in \{1,\dots,r\} \times \mathbb{N}}$ is called a *global system of holomorphic multivalued projections* $X \longrightarrow M$, if for each $j \in \mathbb{N}$ the mapping $\Delta_{s,j} : U_j \longrightarrow \overleftrightarrow{(M \times \mathbb{C})^{k_{s,j}}}$ ($k_{s,j} \in \mathbb{N}$) is a holomorphic multivalued projection (in the sense of Definition 3), $s = 1, \dots r$, having the following properties

(a) $U_j \subset X$ is domain with $U_j \cap M \neq \varnothing$, $U_j \subset U_{j+1}$, $\bigcup_{j \in \mathbb{N}} U_j = X$;
(b) $\lim_{n \to \infty} \sum_{s=1}^r \Xi_{U_n}(x, \Delta_{s,n}) = 1$, $x \in M$.

Then for each $j \in \mathbb{N}$ we get a linear continuous operator $L_{\Pi,j} : \mathcal{O}(M) \longrightarrow \mathcal{O}(U_j)$ such that $\lim_{j \to \infty} L_{\Pi,j}(u)(x) = u(x)$, $x \in M$.

Let $\varnothing \neq \mathcal{F} \subset \mathcal{O}(M)$. We say a global system of holomorphic multivalued projections $\Pi = (\Delta_{s,j})_{(s,j) \in \{1,\dots,r\} \times \mathbb{N}}$ is an $\mathcal{F}$*-extension*, if for each $u \in \mathcal{F}$ the sequence $(L_{\Pi,j}(u))_{j=1}^\infty$ converges locally uniformly in X. We set $L_\Pi(u) := \lim_{j \to \infty} L_{\Pi,j}(u)$, $u \in \mathcal{F}$.

The operator L_Π has the following properties:

(a) $L_\Pi : \mathcal{F} \longrightarrow \mathcal{O}(X)$ is a extension operator.
(b) If $u, v \in \mathcal{F}$ and $u + v \in \mathcal{F}$, then $L_\Pi(u + v) = L_\Pi(u) + L_\Pi(v)$.
(c) If $u \in \mathcal{F}$, $\alpha \in \mathbb{C}$ and $\alpha u \in \mathcal{F}$, then $L_\Pi(\alpha u) = \alpha L_\Pi(u)$.
(d) If $\mathcal{F}$ is vector space, then L_Π is linear.
(e) If $u_1, \dots, u_m \in \mathcal{F}$ are linearly independent (in $\mathcal{O}(M)$), then the formula

$$L_\Pi(\alpha_1 u_1 + \dots + \alpha_m u_m) := \alpha_1 L_\Pi(u_1) + \dots + \alpha_m L_\Pi(u_m), \quad \alpha_1, \dots, \alpha_m \in \mathbb{C},$$

extends the operator L_Π to the vector space $\operatorname{span}\{u_1, \dots, u_m\}$.

We will prove the following theorem.

Theorem 2. *Assume that $\mathcal{F} \subset \mathcal{O}(M)$ is locally uniformly bounded. Then there exists an $\mathcal{F}$-extension $\Pi = (\Delta_{s,j})_{(s,j)\in\{1,\ldots,d\}\times\mathbb{N}}$ with $d := \dim M$. Consequently, there exists a continuous extension operator $L_\Pi : \mathcal{F} \longrightarrow \mathcal{O}(X)$.*

As an application we will get the following result.

Proposition 2. *Assume that $\mathcal{H} \subset \mathcal{O}(M)$ is a Hilbert space such that unit ball $B := \{f \in \mathcal{H} : \|f\|_{\mathcal{H}} \leq 1\}$ is locally uniformly bounded and the convergence in the sense of $\mathcal{H}$ implies the locally uniform convergence in M. Then there exists a linear continuous extension operator $L : \mathcal{H} \longrightarrow \mathcal{O}(X)$. In particular, there exists a linear continuous extension operator $L : L^2_h(M) \longrightarrow \mathcal{O}(X)$.*

Used symbols

$\mathbb{C}$	–	the set of complex numbers,
$\mathbb{N}$	–	the set of natural numbers,
$\mathcal{O}(X)$	–	the set of holomorphic funtions on X,
$L^2_h(X)$	–	$\{f \in \mathcal{O}(X) : \int_X \|f\|^2 < \infty\}$.

References

[1] R. Arens, *Dense inverse limit rings*, Michigan Math. J. 5 (1958), 169-182.

[2] E. Bishop, *Some global problems in the theory of functions of several complex variables*, Amer. J. Math. 83 (1961), 479-498.

[3] K. Drzyzga, *A remark on Bishop's multivalued projections*, Univ. Iag. Acta Math. 53 (2016) (to appear).

[4] R. Gunning, H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Engelewood Cliffs, 1965.

[5] S. Łojasiewicz, *Introduction to Complex Analytic Geometry*, Birkhäuser, 1991.

[6] H. Whitney, *Complex Analytic Varieties*, Addison-Wesley Publishing Company, 1972.