$\mathbb{C}$-CONVEXITY OF SEMITUBE DOMAINS

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Let $\Pi : \mathbb{C}^n \longrightarrow \mathbb{R}^{2n-1}$ be defined by

$$\Pi(z_1, \dots, z_n) := (\operatorname{Re} z_1, \operatorname{Im} z_1, \dots, \operatorname{Re} z_{n-1}, \operatorname{Im} z_{n-1}, \operatorname{Re} z_n).$$

Let $D \subset \mathbb{R}^{2n-1}$ be a domain, $n > 1$. The *semitube domain with the base* D is defined as

$$\mathcal{S}_D := \Pi^{-1}(D).$$

A domain $D \subset \mathbb{C}^n$ is called

- $\mathbb{C}$-*convex*, if for any affine complex line L such that $L \cap D \neq \varnothing$, the set $L \cap D$ is connected and simply connected,
- *linearly convex*, if its complement is a union of affine complex hyperplanes.

Any convex domain is $\mathbb{C}$-convex and any $\mathbb{C}$-convex domain is linearly convex, but the converses do not hold in general.

The aim of this talk is to prove the following result.

Theorem 1. *Let D be a domain in $\mathbb{R}^{2n-1}$ such that $\{x = (x', x_{2n-1}) \in \mathbb{R}^{2n-1} : x' = a'\} \not\subset \partial D$ for any $a \in \partial D$. Then $\mathcal{S}_D$ is $\mathbb{C}$-convex if and only if it is convex.*

Remark 2. (a) If $D = \Omega \times \mathbb{R}$ for some domain $\Omega \subset \mathbb{R}^{2n-2}$ (i.e. $\{x \in \mathbb{R}^{2n-1} : x' = a'\} \subset \partial D$ for any $a \in \partial D$) then the assertion of Theorem 1 is no longer true. Indeed, if Ω is non-convex and $\mathbb{C}$-convex (as a domain considered in $\mathbb{C}^{n-1}$), then $\mathcal{S}_D$ is non-convex but $\mathbb{C}$-convex semitube domain.

(b) Although the condition imposed onto the domain D in Theorem 1 seems to be a technical one, the example in part (a) shows that some restriction of this kind is needed, if we want to have the equivalence of the notions of convexity and $\mathbb{C}$-convexity in the class of semitube domains. It is an open question whether the condition assumed in Theorem 1 is a necessary one for the aforementioned equivalence.

The following observation is crucial in the proof of Theorem 1.

Proposition 3. *Let D be a domain in $\mathbb{R}^{2n-1}$, $n > 1$. Then the following conditions are equivalent:*

(i) $\mathcal{S}_D$ is linearly convex,

(ii) for any $a = (a', a_{2n-1}) \in \mathbb{R}^{2n-1} \setminus D$ there exists affine subspace $H \subset \mathbb{R}^{2n-1}$, $\operatorname{codim}_{\mathbb{R}} H \in \{1, 2\}$, such that $a \in H$, $H \cap D = \varnothing$,

$$H = \begin{cases} \{x \in \mathbb{R}^{2n-1} : b \bullet (x' - a') = \tilde{b} \bullet (x' - a') = 0\}, & \text{if } \operatorname{codim}_{\mathbb{R}} H = 2 \\ \{x \in \mathbb{R}^{2n-1} : x_{2n-1} = a_{2n-1} - b \bullet (x' - a')\}, & \text{if } \operatorname{codim}_{\mathbb{R}} H = 1 \end{cases} \tag{1}$$

for some $b \in \mathbb{R}^{2n-2}$, where $\tilde{b} = (\tilde{b}_1, \dots, \tilde{b}_{2n-2})$,

$$\tilde{b}_j = \begin{cases} -b_{j+1}, & \text{if } j \text{ is odd} \\ b_{j-1}, & \text{if } j \text{ is even} \end{cases}, \quad j = 1, 2, \dots, 2n-2;$$

moreover, if $\operatorname{codim}_{\mathbb{R}} H = 2$, then $b \neq 0$.

For a domain $G \subset \mathbb{C}^n$ and a point $w \in \mathbb{C}^n$, we denote by $\Gamma_G(w)$ the set of all complex hyperplanes L such that $(w + L) \cap G = \varnothing$. One may identify this set with a subset of complex projective space $\mathbb{P}^{n-1}$: here $L = \{z \in \mathbb{C}^n : b \bullet z = 0\}$ is identified with $[b] \in \mathbb{P}^{n-1}$. In the proof of Theorem 1 we shall use the following characterization of $\mathbb{C}$-convexity: if a domain $G \subset \mathbb{C}^n$, $n > 1$, is $\mathbb{C}$-convex then for any $w \in \partial G$ the set $\Gamma_G(w)$ is non-empty and connected.

Let $0 \leq d \leq k$ be two integers. The Grassmann manifold $\operatorname{Gr}(d, \mathbb{R}^k)$ is the set of all d-dimensional real subspaces of $\mathbb{R}^k$ which is topologized as a quotient space. In the proof of Theorem 1 we shall use the fact that the Grassmann manifold $\operatorname{Gr}(d, \mathbb{R}^k)$ is compact.