Bergman kernel and hyperconvexity index (based on paper by B.-Y. Chen)

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Given a domain $\Omega \subset \mathbb{C}^n$ denote its Bergman kernel by $K_\Omega(z,w)$ and let h be a (negative) relative extremal function of some fixed closed ball $B \subset \Omega$ with respect to Ω. We present main results from the paper *Bergman kernel and hyperconvexity index* by B.-Y. Chen (Analysis and PDE, 10 (6), 2017) concerning the integrability as well as certain estimates of Bergman kernels, which reads as follows:

Theorem. *Let $\Omega \subset\subset \mathbb{C}^n$ be such that $\alpha(\Omega) > 0$, where*

$$\alpha(\Omega) := \sup\{\alpha > 0 : \exists C > 0, \rho \in \mathcal{PSH}^-(\Omega) \cap \mathcal{C}(\Omega) \ s.t. \ -\rho \leq C\delta^\alpha\}$$

is the so called hyperconvexity index (here δ denotes the distance to the boundary). Then:

(1) *For every $\alpha \in (0, \alpha(\Omega))$ and every $p \in [2, 2 + 2\alpha/(2n - \alpha))$ there exists a $C > 0$ such that for any $w \in \Omega$ we have*

$$\int_\Omega |K_\Omega(\cdot, w)|^p \leq C(\sqrt{K_\Omega}(w,w))^p(|h(w)|(1 + |\log|h(w)||)^{-1})^{(2-p)n/\alpha}.$$

(2) *For every $r \in (0,1)$ there exists a $C > 0$ such that for any $z, w \in \Omega$ we have*

$$\frac{|K_\Omega(z,w)|^2}{K_\Omega(z,z)K_\Omega(w,w)} \leq C\left(\min\left\{\frac{|h(w)|(1 + |\log|h(w)||)^n}{|h(z)|(1 + |\log|h(z)||)^{-1}}, \frac{|h(z)|(1 + |\log|h(z)||)^n}{|h(w)|(1 + |\log|h(w)||)^{-1}}\right\}\right)^r.$$

We discuss the applications of the above estimates, also given in said paper, for example the following estimate for the boundary behaviour of Bergman distance $\mathcal{B}_\Omega$:

Theorem. *Let $\Omega \subset\subset \mathbb{C}^n$ be such that $\alpha(\Omega) > 0$ and let $z_0 \in \Omega$. Then there exists a $C > 0$ such that*

$$\mathcal{B}_\Omega(z_0, z) \geq C\frac{|\log\delta(z)|}{\log|\log\delta(z)|},$$

if only z is sufficiently close to $\partial\Omega$.

This kind of estimate was first obtained by Z. Błocki for bounded domains admitting a continuous negative plurisubharmonic function ρ satisfying $C_1\delta^\alpha \leq -\rho \leq C_2\delta^\alpha$ with some positive constants $C_1, C_2, \alpha > 0$ (cf. Z. Błocki, *The Bergman metric and the pluricomplex Green function*, Trans. Amer. Math. Soc. 357:7 (2005), 2613-2625).