Dense Analytic Curves Generated by Iterations of Complex Periodic Functions

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Abstract

The present paper is devoted to finding the classes of complex periodic functions whose iterations map lines into dense subsets of the complex plane. We distinguish a class of functions such that its second iterations map almost all oblique lines into dense sets. There is found a class of complex periodic functions whose third iterations of any oblique line generate dense subsets of the complex plane. Finally, we establish connection between density of high order iterations of an arbitrary subset in the complex plane.

Keywords:
Complex periodic functions, Iterations, Dense sets, Distortion

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1. Introduction

In 1914, H. Bohr and R. Courant showed that the image of any vertical line with real part in $(1/2, 1]$ is dense in $\mathbb{C}$ under the Riemann zeta function [1]. Basing on the Birkhoff result which states the existence of an entire function $f : \mathbb{C} \to \mathbb{C}$ whose translates $T_n f : x \mapsto f(x-n)$ approximate polynomials in $\mathbb{Q}[x] + i\mathbb{Q}[x]$ locally uniformly [2], N.Dobbs has suggested a construction of a countable family of entire functions which map the real line into dense subsets of the complex plain [3]. As was mentioned in [3], there is no a reasonable way to check whether a particular function belongs to this family. N. Dobbs has investigated

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the problem of generating of dense analytic curves for the exponential mapping iterations: there was proved that the function $\exp \circ \exp$ maps almost all oblique lines into dense sets of $\mathbb{C}$ and the function $\exp \circ \exp \circ \exp$ maps all oblique lines into dense subsets of $\mathbb{C}$ [3]. In the present paper we construct classes of analytic functions which generate dense curves in $\mathbb{C}$. Verification of belonging of particular functions to our classes comes down to some routine procedures (see definitions of classes $\mathcal{A}$ and $\mathcal{B}$ in Section 2). While developing the ideas of the paper [3] for the exponential map, we find a class of analytic periodic functions whose second iterations map almost all oblique lines into dense sets of $\mathbb{C}$ (see Theorem 1 in Section 3) and obtain the class of analytic periodic functions whose third iterations map any oblique line into a dense subset of the complex plane (see Theorem 2 in Section 4).

Let us recall a concept of distortion and the Lebesgue density point theorem which will play an important role in our investigations.

Let $f : A \to \mathbb{C}$ be an analytic non-zero function defined on a non-empty subset $A \subseteq \mathbb{C}$. For any non-empty subset $B \subseteq A$ define distortion of the function f on the set B as

$$D(f, B) = \frac{\sup\limits_{z \in B} |f'(z)|}{\inf\limits_{z \in B} |f'(z)|}$$

if the right-hand side is well-defined [3], [4, P. 263].

Let $X \subset \mathbb{R}$ be a Lebesgue measurable set, $\varepsilon > 0$. Recall that a point $x \in \mathbb{R}$ is an ε-density point for a set $X \subset \mathbb{R}$ if

$$\lim_{r \to +0} \frac{m(X \cap B(x, r))}{m(B(x, r))} \geq \varepsilon,$$

where $m(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}$, $B(x, r)$ is the open ball in $\mathbb{R}$ with the center in x and radius r. An open ball in $\mathbb{C}$ with center in $x \in \mathbb{C}$ of radius r will be also denoted by $B(x, r)$.

By the Lebesgue density point theorem, almost every point of a measurable set $X \subset \mathbb{R}$ is a 1-density point for X [5, Chapter 1].

For $p \in \mathbb{C}$, $\alpha \in \mathbb{R}$, let $L_\alpha(p) := \{p + t(i + \alpha) : t \in \mathbb{R}\}$. Denote by $f^{\circ(n)}$ the n-multiple composition of the function $f : \mathbb{C} \to \mathbb{C}$.

2. Definition of classes $\mathcal{A}$ and $\mathcal{B}$

We will say that an analytic function $f : \mathbb{C} \to \mathbb{C}$ belongs to **class** $\mathcal{A}$ if for some $j \in \{1, i\}$ the following conditions hold:

$(\mathcal{A}_1)$: the function f is Tj-periodic for some real positive number T;

$(\mathcal{A}_2)$: there exists a finite set $H = \{h_1, \dots, h_m\}$ of real numbers such that one can define an inverse analytic single-valued function $\lambda(z) = f^{-1}(z)$ on the set $\mathbb{C} \setminus E$, where $E \subset \mathbb{C}$ is the set of all points which belong to the lines $z = (t + ih_k)j$, $t \in \mathbb{R}$, $k = 1, \dots, m$;

$(\mathcal{A}_3)$: for any real $h \notin H$ there exists a constant $c(h)$ such that

$$\sup_{t\in\mathbb{R}} |\lambda'((t+ih)j)| \le c(h);$$

$(\mathcal{A}_4)$: for any positive ε and h there exists a constant $d(\varepsilon, h)$ such that the inequality

$$D(f, S_{a,b}) \le d(\varepsilon, h)$$

holds for any real a, b such that $a \le b \le a+h$, $\min\limits_{\xi\in[a,b],\eta\in H} |\xi-\eta| \ge \varepsilon$, where $S_{a,b} = \{(t+i\zeta)j \in \mathbb{C} : \zeta \in [a,b], t \in \mathbb{R}\}$;

$(\mathcal{A}_5)$: for any fixed real $h \notin H$ the function $\varphi(t)$, $t \in \mathbb{R}$, is bounded, and $\frac{\psi'(t)}{\varphi'(t)} \to \infty$ as $t \to \infty$, where $\varphi(t) = \mathrm{Re}(j\lambda((t+ih)j))$, $\psi(t) = \mathrm{Im}(j\lambda((t+ih)j))$.

Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function. We will say that the function f belongs to **class** $\mathcal{B}$ if there hold conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_5)$ and, additionally, the following conditions (here we will use notation from class $\mathcal{A}$ definition):

$(\mathcal{B}_1)$: for any real $h \notin H$ there exists a positive real number ε such that

$$\lim_{t\to\infty} \sup_{\zeta\in[h-\varepsilon,h+\varepsilon]} |\lambda'((t+i\zeta)j)| = 0;$$

$(\mathcal{B}_2)$: for any $p \in \mathbb{C}$, $\alpha \in \mathbb{R} \setminus \{0\}$, $\delta > 0$, $l > 0$ there exists a segment $[t_1, t_2]$ such that length of the curve segment $\eta(t) = f(p + t(i+\alpha))$, $t \in [t_1, t_2]$, is at least l, and the slope at $\eta(t)$ is at least δ and at most 2δ for each $t \in [t_1, t_2]$.

In Sections 3 and 4 we demonstrate that the mappings $f(z) = \mathrm{e}^z$ and $f(z) = \sin(z)$ belong to classes $\mathcal{A}$ and $\mathcal{B}$ (see Examples 1, 2, 3, 4).

3. Dense analytic curves generated by second iterations

In this section we are going to prove that for any function f from class $\mathcal{A}$ its second iteration $f^{\circ(2)}$ maps almost all oblique lines into dense sets of the complex plane.

Theorem 1. *Let $p \in \mathbb{C}$, and let a function $f : \mathbb{C} \to \mathbb{C}$ be from class $\mathcal{A}$. Then for almost all $\alpha \in \mathbb{R}$ the set $f^{\circ(2)}(L_\alpha(p))$ is dense in $\mathbb{C}$.*

Proof. We denote $g = f \circ f$, fix an arbitrary $p \in \mathbb{C}$ and write L_α instead of $L_\alpha(p)$. For an arbitrary open ball $U \subset \mathbb{C}$ we define the set

$$X_U = \{\alpha : g(L_\alpha) \cap U \ne \varnothing\}.$$

Let's prove that for any open ball $U \subset \mathbb{C}$ the set X_U is Lebesgue measurable. Define the multivalued mapping $q : \mathbb{R} \to \mathrm{cl}(\mathbb{C})$ as follows:

$$q(\alpha) = \{p + t(i+\alpha) : t \in \mathbb{R}\},$$

where $\mathrm{cl}(\mathbb{C})$ is the set of all non-empty closed subsets of $\mathbb{C}$. It is clear that the mapping q is continuous. Since the function f is analytic, the mapping $z \to g(z)$ is a Caratheodory mapping. Hence, the multivalued function $\alpha \to g(q(\alpha))$ is measurable [6, Lemma 8.2.3]. That is why the set X_U is also measurable.

We proceed with the following statement.

Proposition 1. *If for any open ball $U \subset \mathbb{C}$ with $\partial U \cap E = \varnothing$ there exists $\varepsilon > 0$ such that any $\alpha \in \mathbb{R} \setminus \{0\}$ is an ε-density point for X_U, then for almost all $\alpha \in \mathbb{R}$ the set $g(L_\alpha)$ is dense in $\mathbb{C}$.*

PROOF. We take a sequence $(q_n)_{n=1}^{\infty}$ dense in $\mathbb{C}$ and a decreasing sequence $(\delta_n)_{n=1}^{\infty}$ of positive real numbers with $\delta_n \to 0$ as $n \to \infty$, such that for any $n \in \mathbb{N}$ the condition $\partial U_n \cap E = \varnothing$ holds, where $U_n \subset \mathbb{C}$ is an open ball of radius δ_n with center in q_n. It is clear that the set $g(L_\alpha)$ is dense in $\mathbb{C}$ if and only if for any $n \in \mathbb{N}$ the set $g(L_\alpha) \cap U_n$ is non-empty. Let

$$X_\infty = \bigcap_{n=1}^{\infty} X_{U_n}.$$

Since for any $\alpha \in X_\infty$ the set $g(L_\alpha)$ is dense in $\mathbb{C}$, we only need to show that for any $n \in \mathbb{N}$ the set X_{U_n} has full measure.

Let's prove that for any $n \in \mathbb{N}$ the set $Y_n = \mathbb{R} \setminus X_{U_n}$ has zero measure. Since for any $n \in \mathbb{N}$ there exists $\varepsilon_n > 0$ such that any $\alpha \in \mathbb{R} \setminus \{0\}$ is an ε_n-density point for the set X_{U_n}, the set of 1-density points for Y_n has zero measure. Hence, by the Lebesgue density point theorem, Y_n has zero measure. Therefore, for any $n \in \mathbb{N}$ the set X_{U_n} has full measure. The proposition is proved. $\square$

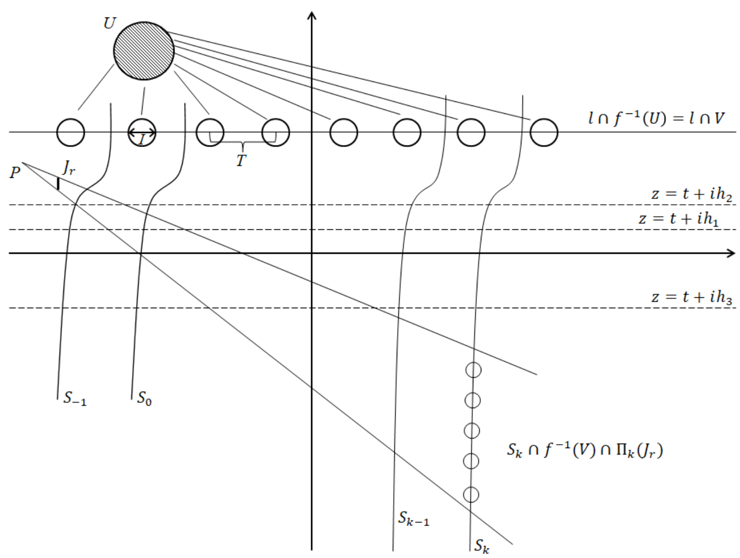

Figure 1: Intersection of $L_\alpha(p)$ with $f^{-1}(f^{-1}(U))$ in case $j = 1$

We get arbitrary open ball $U \subset \mathbb{C}$ with $\partial U \cap E = \varnothing$. Since the function $\lambda(z)$ is well-defined and analytic on ∂U, we obtain that $\lambda(\partial U)$ is a differentiable simple closed curve bounding a simply connected set D. Consider a line l given as $z = (t + ih)j$, $t \in \mathbb{R}$, $h \notin H$, such that the set $l \cap D$ contains an open interval I. Let $V = f^{-1}(U)$, $S = f^{-1}(l)$. It's clear that the set $l \cap V$ contains the interval

I and all its Tjk-translates, $k \in \mathbb{Z}$. For any subinterval $\gamma \subset l$ with length at least T there holds the inequality

$$\frac{m(\gamma \cap V)}{m(\gamma)} \geq \frac{m(I)}{2T}. \tag{1}$$

Denote by S_0 the connected component of the set S corresponding to the branch $\lambda(z)$ of the multivalued inverse function $f^{-1}(z)$. All other connected components can be obtained from S_0 by translations: $S_k = S_0 + Tjk$, $k \in \mathbb{Z}$. Let $W_k = S_k \cap f^{-1}(V)$.

Since $\lambda(S_0) \subset l$, condition $(\mathcal{A}_3)$ implies that $|f'(z)| \geq 1/c(h)$ for any $z \in S_0$. That is why any segment of S_0 (and, hence, S_k) of length at least $T/c(h)$ is mapped by $f(z)$ onto a segment of l of length at least T.

Let

$$\rho = \max_{k=\overline{1,n}} |h_k| + 1.$$

Let $h_1 = 2T/c(h)$. By assumption $(\mathcal{A}_4)$, for any segment B of S_k of length between $T/c(h)$ and h_1, which belongs to the set

$$Z_\rho = \{(t + i\zeta)j \in \mathbb{C} : t, \zeta \in \mathbb{R}, |\zeta| \geq \rho\},$$

there holds the inequality

$$\frac{m(B \cap W_k)}{m(B)} \geq \frac{m(f(B) \cap V)}{m(f(B))d(1, h_1)}. \tag{2}$$

Combining relations (1) and (2), we get the inequality

$$\frac{m(B \cap W_k)}{m(B)} \geq \frac{m(I)}{2Td(1, h_1)}. \tag{3}$$

We are going to show that the restriction on upper boundedness of a segment B can be omitted. Indeed, let B be a segment of S_k of length at least $T/c(h)$ which belongs to the set Z_ρ. Split the segment B into pairwise disjoint (except the ending points) segments $B_1, \ldots, B_s$ of lengths between $T/c(h)$ and h_1. Inequality (3) remains valid after replacing B by any B_i, $i = 1, \ldots, s$. That is why relation (3) holds for any segments B of S_k of lengths at least $T/c(h)$ which belong to the set Z_ρ.

Let $\alpha_0 \in \mathbb{R} \setminus \{0\}$, $r_0 = |\alpha_0|/2$. Denote by J_{r_0} the line segment $z = p + j + ij(\alpha_0 + r)$, $r \in [-r_0, r_0]$.

If $j = 1$, then the curve S_0 has parametric equation $z = \lambda(t + ih) = \varphi(t) + i\psi(t)$, $t \in \mathbb{R}$. Since, by assumption $(\mathcal{A}_5)$, the function $\varphi(t)$ is bounded and $\frac{\psi'(t)}{\varphi'(t)} \to \infty$ as $t \to \infty$, the curve S_0 has a vertical asymptote. Analogously, if $j = i$, the curve S_0 has parametric equation $z = \lambda(-h + it) = \psi(t) - i\varphi(t)$, $t \in \mathbb{R}$, and there exists a horizontal asymptote for S_0.

It is easy to see that one can find a constant $K \in \mathbb{N}$ such that for any $k \geq K$ and any $\alpha \in [\alpha_0 - r_0, \alpha_0 + r_0]$ the line L_α intersects the curve S_k. For any $k \geq K$ define a function $\Pi_k : [-r_0, r_0] \to \mathbb{C}$ as follows: for any fixed $r \in [-r_0, r_0]$ let $\Pi_k(r)$ be the nearest to p point of intersection of the line l_r and the curve S_k, where l_r is the line which passes through the points p and $p + j + ij(\alpha_0 + r)$.

Proposition 2. *There holds the equality*

$$\lim_{k\to\infty} D(\Pi_k, [-r_0, r_0]) = 1.$$

Proof. Let us consider the case $j = 1$. The curve S_k has parametric equation $z = \varphi(t) + Tk + i\psi(t)$. Let $\Pi_k(r) = \varphi(t_{k,r}) + Tk + i\psi(t_{k,r})$, where $k \geq K$, $r \in [-r_0, r_0]$. It is easy to see that there holds the equality

$$\lim_{k\to\infty} \inf_{r\in[-r_0,r_0]} |t_{k,r}| = \infty. \tag{4}$$

Let $p_1 = \mathrm{Re}(p)$, $p_2 = \mathrm{Im}(p)$, then for any $r \in [-r_0, r_0]$ equation of the line l_r has the form:

$$\frac{x - p_1}{1} = \frac{y - p_2}{\alpha_0 + r}.$$

We fix an arbitrary $k \geq K$. Denote $x(r) = \mathrm{Re}(\Pi_k(r))$, $y(r) = \mathrm{Im}(\Pi_k(r))$, then the function $X = (x, y) : [-r_0, r_0] \to \mathbb{R}^2$ is an implicit function defined by the system of equations

$$\begin{cases} x - \varphi(t) - Tk = 0, \\ y - \psi(t) = 0, \\ y - p_2 - (x - p_1)(\alpha_0 + r) = 0. \end{cases}$$

Denote $Y = (x, y, t)$. So we can rewrite our system as $F(Y, r) = 0$ and consider the implicit vector function $Y = G(r)$ with $x = g_1(r)$, $y = g_2(r)$, $t = g_3(r)$. Since $\frac{\psi'(t)}{\varphi'(t)} \to \infty$ as $t \to \infty$, the determinant of the matrix

$$\frac{\partial F}{\partial Y} = \begin{pmatrix} 1 & 0 & -\varphi'(t) \\ 0 & 1 & -\psi'(t) \\ -\alpha_0 - r & 1 & 0 \end{pmatrix}$$

does not vanish for any $r \in [-r_0, r_0]$ and for all big enough t. The implicit function theorem implies that

$$\frac{\partial G}{\partial r} = -\left(\frac{\partial F}{\partial Y}\right)^{-1} \frac{\partial F}{\partial r}.$$

Since

$$\frac{\partial F}{\partial r} = \begin{pmatrix} 0 \\ 0 \\ p_1 - x \end{pmatrix},$$

we get that the equalities

$$g_1'(r) = \frac{x - p_1}{-\alpha_0 - r + \frac{\psi'(t)}{\varphi'(t)}} = \frac{\varphi(t) + Tk - p_1}{-\alpha_0 - r + \frac{\psi'(t)}{\varphi'(t)}},$$

$$g_2'(r) = \frac{x - p_1}{-\alpha_0 - r + \frac{\psi'(t)}{\varphi'(t)}} \frac{\psi'(t)}{\varphi'(t)} = \frac{\varphi(t) + Tk - p_1}{-\alpha_0 - r + \frac{\psi'(t)}{\varphi'(t)}} \frac{\psi'(t)}{\varphi'(t)}$$

hold for any $r \in [-r_0, r_0]$ and for all big enough t.

Taking into account assumption $(\mathcal{A}_5)$ and relation (4), we obtain

$$\sup_{r\in[-r_0,r_0]} |\Pi_k'(r)|^2$$

$$= \sup_{r\in[-r_0,r_0]} \left(\left(\frac{\varphi(t)+Tk-p_1}{-\alpha_0 - r + \frac{\psi'(t)}{\varphi'(t)}} \right)^2 + \left(\frac{\varphi(t)+Tk-p_1}{-\alpha_0 - r + \frac{\psi'(t)}{\varphi'(t)}} \frac{\psi'(t)}{\varphi'(t)} \right)^2 \right)$$

$$\sim \sup_{r\in[-r_0,r_0]} \left(\frac{\frac{\psi'(t)}{\varphi'(t)}}{-\alpha_0 - r + \frac{\psi'(t)}{\varphi'(t)}} \right)^2 (\varphi(t)+Tk-p_1)^2 \sim T^2k^2,$$

as $k \to \infty$. Analogously we get

$$\inf_{r\in[-r_0,r_0]} |\Pi_k'(r)|^2 \sim T^2k^2$$

as $k \to \infty$. Hence, the proposition is proved for $j = 1$. The case $j = i$ is considered analogously. $\square$

For any $r \in (0, r_0)$ there exists a positive integer $k_r \geq K$ such that

$$\Pi_{k_r}(J_r) \subset Z_\rho, \tag{5}$$

$$m(\Pi_{k_r}(J_r)) > T/c(h). \tag{6}$$

Denote $X_r = J_r \cap \Pi_{k_r}^{-1}(W_{k_r})$. It follows from relations (3), (5), (6) and Proposition 2 that for any $r \in (0, r_0)$ there hold the inequalities

$$\frac{m(X_r)}{m(J_r)} \geq \frac{m(W_{k_r} \cap \Pi_{k_r}(J_r))}{2m(\Pi_{k_r}(J_r))} \geq \frac{m(I)}{4Td(1,h_1)}. \tag{7}$$

Denote by Y_r the set of all $\alpha \in \mathbb{R}\setminus\{0\}$ such that $p+j+ij(\alpha+r) \in X_r$, then for any $\alpha \in Y_r$ there holds $L_\alpha \cap W_{k_r} \neq \varnothing$, and hence $g(L_\alpha) \cap U \neq \varnothing$. That is why $Y_r \subseteq X_U$ for any $r \in (0, r_0)$. Taking into account relation (7), we get that the inequalities

$$\frac{m(X_U)}{m(B(\alpha_0,r))} \geq \frac{m(Y_r)}{m(B(\alpha_0,r))} = \frac{m(X_r)}{m(J_r)} \geq \frac{m(I)}{4Td(1,h_1)}$$

hold for any $r \in (0, r_0)$. Hence, by Proposition 1, we conclude that the set $f^{\circ(2)}(L_\alpha(p))$ is dense in $\mathbb{C}$. The theorem is proved. $\square$

Example 1. Let us demonstrate that the complex exponent $f(z) = \mathrm{e}^z$ belongs to class $\mathcal{A}$. Take $j = i$, $T = 2\pi$, $H = \{0\}$, $\lambda(z) = \mathrm{Ln}(z)$.

Let's verify the validity of assumption $(\mathcal{A}_3)$. Indeed,

$$\sup_{t\in\mathbb{R}} |\lambda'(h+it)| = \sup_{t\in\mathbb{R}} \left| \frac{1}{h+it} \right| = \frac{1}{|h|}$$

for any $h \neq 0$.

Let us check condition $(\mathcal{A}_4)$. Take arbitrary $\varepsilon > 0$, $h > 0$ and $a, b \in \mathbb{R}$, $a \leq b \leq a + h$, $\min\limits_{\xi \in [a,b]} |\xi| \geq \varepsilon$, $S_{a,b}$ is the vertical stretch $a \leq \mathrm{Re}(z) \leq b$. We get

$$D(f, S_{a,b}) = \sup_{z_1, z_2 \in S_{a,b}} |\mathrm{e}^{z_1 - z_2}| \leq \mathrm{e}^{b-a} \leq \mathrm{e}^h.$$

Let us check condition $(\mathcal{A}_5)$. Let $h \in \mathbb{R} \setminus \{0\}$. Then

$$\lambda(h + it) = \frac{1}{2} \log(t^2 + h^2) + i \arctan \frac{t}{h},$$

so the function $\varphi(t) = -\arctan \frac{t}{h}$ is bounded, and $\frac{\psi'(t)}{\varphi'(t)} = -\frac{t}{h} \to \infty$ as $t \to \infty$, where $\psi(t) = \frac{1}{2} \log(t^2 + h^2)$.

Example 2. Let us show that the complex sinus $f(z) = \sin z$ belongs to class $\mathcal{A}$. Take $j = 1$, $T = 2\pi$, $H = \{0\}$, $\lambda(z) = \mathrm{Arcsin}(z)$.

Let's verify the validity of assumption $(\mathcal{A}_3)$. Indeed,

$$\sup_{t \in \mathbb{R}} |\lambda'(t + hi)| = \sup_{t \in \mathbb{R}} \left| \frac{1}{\sqrt{1 - (t + hi)^2}} \right| = \sup_{t \in \mathbb{R}} \frac{1}{\sqrt[4]{(1 - t^2 + h^2)^2 + 4h^2t^2}} \leq \frac{1}{|h|}$$

for any $h \neq 0$.

Let us check condition $(\mathcal{A}_4)$. Take an arbitrary $\varepsilon > 0$, $h > 0$ and $a, b \in \mathbb{R}$, $a \leq b \leq a + h$, $\min\limits_{\xi \in [a,b]} |\xi| \geq \varepsilon$, $S_{a,b}$ is the horizontal stretch $a \leq \mathrm{Im}(z) \leq b$.

Without loss of generality we may assume that $a \geq \varepsilon$. Taking into account the inequality

$$\frac{|\mathrm{e}^y - \mathrm{e}^{-y}|}{2} \leq |\cos(x + iy)| \leq \frac{\mathrm{e}^y + \mathrm{e}^{-y}}{2},$$

which is valid for any x, $y \in \mathbb{R}$, we estimate

$$D(f, S_{a,b}) = \sup_{z_1, z_2 \in S_{a,b}} \left| \frac{\cos z_1}{\cos z_2} \right| \leq \frac{\mathrm{e}^{a+h} + \mathrm{e}^{-(a+h)}}{\mathrm{e}^a - \mathrm{e}^{-a}} \leq \frac{\mathrm{e}^h + 1}{1 - \mathrm{e}^{-2\varepsilon}}.$$

Let us check condition $(\mathcal{A}_5)$. Let $h \in \mathbb{R} \setminus \{0\}$. Then

$$\lambda'(t + hi) = \frac{1}{\sqrt{1 - (t + hi)^2}} = \frac{1}{\sqrt[4]{(1 - t^2 + h^2)^2 + 4h^2t^2}} \left(\cos \frac{\varphi}{2} - i \sin \frac{\varphi}{2} \right),$$

where

$$\cos \varphi = \frac{1 - t^2 + h^2}{\sqrt{(1 - t^2 + h^2)^2 + 4h^2t^2}} \to -1$$

as $t \to \infty$. Since $\lambda'(t + hi) = \varphi'(t) + i\psi'(t)$, we obtain that $\frac{\psi'(t)}{\varphi'(t)} = -\tan \frac{\varphi}{2} \to \infty$ as $t \to \infty$.

It is clear that the function

$$\varphi(t + hi) = \mathrm{Re}(\mathrm{Arcsin}(t + hi))$$

$$= \arctan \frac{t + \sqrt{(1-t^2+h^2)^2 + 4h^2t^2}\cos(\varphi/2)}{-h + \sqrt{(1-t^2+h^2)^2 + 4h^2t^2}\sin(\varphi/2)}, \ t \in \mathbb{R},$$

is bounded. So, $f(z) = \sin(z)$ belongs to class $\mathcal{A}$.

Remark 1. We cannot extend Theorem 1 on all oblique lines $L_\alpha(p)$. Indeed, as was shown in Dobbs (2015), for $f(z) = \mathrm{e}^z$, for each $p \in \mathbb{C}$ and each open set $X \subseteq \mathbb{R}$ the set of all $\alpha \in X$ such that the set $f^{\circ(2)}(L_\alpha(p))$ is not dense in $\mathbb{C}$ has Hausdorff dimension 1.

4. Dense analytic curves generated by high order iterations

In this section we will prove that for a function $f : \mathbb{C} \to \mathbb{C}$ from class $\mathcal{B}$ its third order iteration $f^{\circ(3)}$ maps all oblique lines into dense sets of the complex plane.

Theorem 2. *Let $p \in \mathbb{C}$, a function $f : \mathbb{C} \to \mathbb{C}$ be from class $\mathcal{B}$. Then for any $\alpha \in \mathbb{R} \setminus \{0\}$ the set $f^{\circ(3)}(L_\alpha(p))$ is dense in $\mathbb{C}$.*

Proof. We will use the same notation as was introduced in the proof of Theorem 1. Take an arbitrary open ball $U \subset \mathbb{C}$ with $\partial U \cap E = \varnothing$, then $z = \lambda(\partial U)$ is a differentiable simple closed curve bounding a simply connected set D. Denote $V = f^{-1}(U)$, $S = f^{-1}(l)$, where l is a line $z = (t+ih)j$, $t \in \mathbb{R}$, $h \notin H$, such that the set $l \cap D$ contains an open interval I. The set S consists of the sequence of connected components S_k, $k \in \mathbb{Z}$. Take a point $v_0 \in l \cap D$. Let $v_m = v_0 + Tjm$, $m \in \mathbb{Z}$. It's clear that $v_m \in l \cap V$ for any $m \in \mathbb{Z}$. Let $w_m^k = f^{-1}(\{v_m\}) \cap S_k$, k, $m \in \mathbb{Z}$. It is easy to see that the number $\mathrm{Re}(jw_m^k)$ does not depend on k, denote $\theta_m = \mathrm{Re}(jw_m^k)$. It follows from condition $(\mathcal{A}_5)$ that $\theta_m \to +\infty$ or $-\infty$ as $m \to \infty$. Let Z_m^k be the line segment joining w_m^k and w_{m+1}^k, then the slope of Z_m^k tends to infinity if $j = 1$, and tends to zero if $j = i$. Since l is a horizontal line if $j = 1$, and a vertical line if $j = i$, S_k lies in a vertical strip of width $T/2$ if $j = 1$, and lies in a horizontal strip of height $T/2$ if $j = i$. For any fixed $m_0 \in \mathbb{N}$ define a curve

$$\xi_{m_0}^k = \bigcup_{m \geq m_0} Z_m^k,$$

contained in a vertical (if $j = 1$) or horizontal (if $j = i$) strip of width $T/2$, joining $w_{m_0}^k$ to ∞.

There exists $r \in (0,1)$ such that for any $m \in \mathbb{Z}$ there holds the inclusion $B(v_m, r) \subset V$. Due to condition $(\mathcal{B}_1)$, there exist constants $\beta \in (0,1)$ and $m_0 \in \mathbb{N}$ such that for any $m \geq m_0$ and $k \in \mathbb{Z}$ there hold the relations

$$B(w_m^k, \beta r) \subset f^{-1}(V), \tag{8}$$

$$|w_{m+1}^k - w_m^k| \leq T\beta. \tag{9}$$

Let $\delta = \beta r/T$, then relations (8) and (9) imply that for any $m \geq m_0$ and $k \in \mathbb{Z}$ there holds the inclusion

$$B(w_m^k, \delta|w_{m+1}^k - w_m^k|) \subset f^{-1}(V).$$

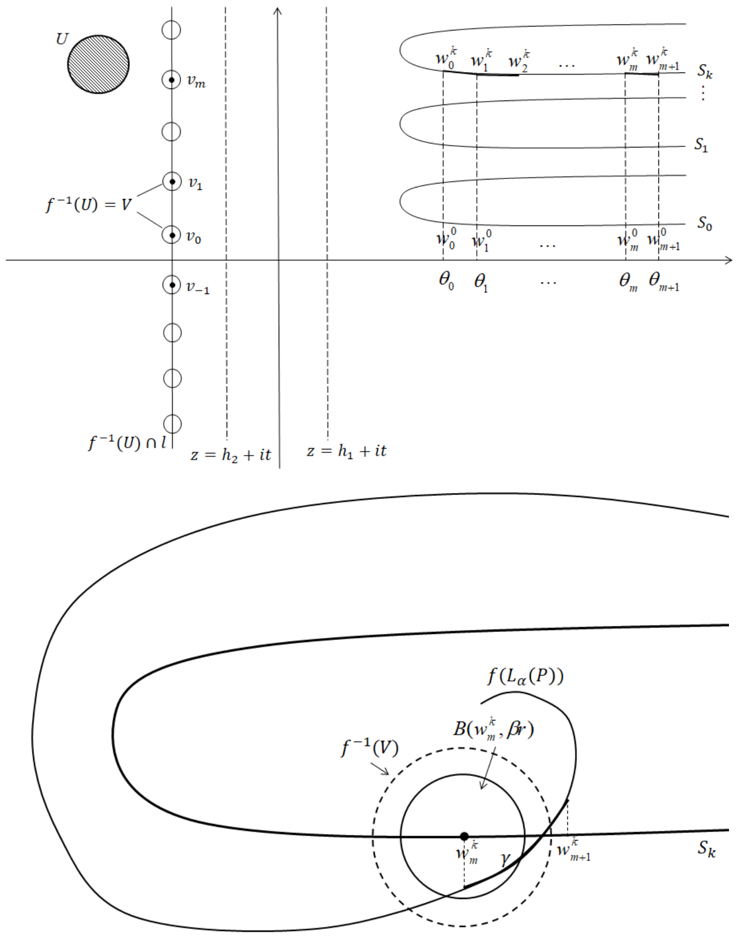

Figure 2: Intersection of $f(L_\alpha(p))$ with $f^{-1}(f^{-1}(U))$ in case $j = i$

Let's consider the case $j = i$. Take an arbitrary $\varepsilon > 0$. It follows from condition $(\mathcal{A}_5)$ that there exists a number $m'(\varepsilon) \geq m_0$ such that the absolute value of slope at $\xi^k_{m'(\varepsilon)}$ is upper bounded by ε. Due to condition $(\mathcal{B}_2)$, there exists a curve segment γ given as $\eta(t) = f(p + t(i + \alpha))$, $t \in [t_1, t_2]$, whose slope belongs to $(\delta/4, \delta/2)$ at each point $t \in [t_1, t_2]$, such that the segment γ lies in the set

$$\{z \in \mathbb{C} : |\mathrm{Re}(z)| > \theta_{m'(\varepsilon)}\},$$

and have the projection on the x-axis of length at least $l \geq 2T + 6T/\delta$.

It follows from relation (9) that one can find a subcurve γ_1 of the curve γ such that the projection of γ_1 on the x-axis is $[\theta_{m_1}, \theta_{m_2}]$ for some m_1, $m_2 \in \mathbb{N}$, and $\theta_{m_2} - \theta_{m_1} \geq 6T/\delta$. Since the slope at γ_1 is lower bounded by $\delta/4$, the range of the imaginary part of γ_1 is at least $3T/2$. That is why the curve γ_1 (and, hence, the curve γ) intersects the curve $\xi^k_{m'(\varepsilon)}$ for some $k \in \mathbb{Z}$. Let the curve γ intersect Z^k_m. Since the slope at γ_1 is bounded in absolute value by $\delta/2$ and the slope at Z^k_m bounded in absolute value by ε, the number ε can be chosen small enough for the curve γ to intersect the ball $B(w^k_m, \delta|w^k_{m+1} - w^k_m|)$. That is why the set $f^{\circ(3)}(L_\alpha(p))$ intersects the ball U.

The case $j = 1$ can be considered similarly. The theorem is proved. $\square$

Example 3. Let us show that the function $f(z) = \mathrm{e}^z$ is from class $\mathcal{B}$.

Since $\lambda'(z) = \frac{1}{z}$, the equality in condition $(\mathcal{B}_1)$ holds for any $h \neq 0$ with $\varepsilon = |h|/2$.

Let us check condition $(\mathcal{B}_2)$. Take an arbitrary $p = p_1 + ip_2 \in \mathbb{C}$, $\alpha \in \mathbb{R} \setminus \{0\}$, $\delta > 0$, $l > 0$. We obtain $f(L_\alpha(p)) = \{x(t) + iy(t) : t \in \mathbb{R}\}$, where

$$x(t) = \exp(p_1 + \alpha t)\cos(p_2 + t),$$

$$y(t) = \exp(p_1 + \alpha t)\sin(p_2 + t).$$

Without loss of generality we may assume that $\alpha > 0$ (otherwise we can replace t by $-t$). Since

$$y'_x(t) = \frac{\alpha\tan(p_2 + t) - 1}{\alpha - \tan(p_2 + t)},$$

the curve $\eta_k = x(t) + iy(t)$, $t \in [t'_k, t''_k]$, has slope between δ and 2δ, where

$$t'_k = \arctan\frac{\delta\alpha}{\alpha + \delta} + \pi k - p_2,\ t''_k = \arctan\frac{2\delta\alpha}{\alpha + 2\delta} + \pi k - p_2.$$

Then

$$|\eta_k| = \int\limits_{t'_k}^{t''_k} \sqrt{(x'(t))^2 + (y'(t))^2}dt \geq \sqrt{\alpha^2 + 1}\exp(p_1 + \alpha t'_k)(t''_k - t'_k) \to \infty$$

as $k \to +\infty$. Hence, there exists $k \in \mathbb{N}$ such that $|\eta_k| \geq l$.

Example 4. Let us show that the function $f(z) = \sin(z)$ also belongs to class $\mathcal{B}$.

Since $\lambda'(t + hi) = \frac{1}{\sqrt{1-(t+hi)^2}}$, the equality in condition $(\mathcal{B}_1)$ holds for any $h \neq 0$ with $\varepsilon = |h|/2$.

Let us check condition $(\mathcal{B}_2)$. Take an arbitrary $p = p_1 + ip_2 \in \mathbb{C}$, $\alpha \in \mathbb{R} \setminus \{0\}$, $\delta > 0$, $l > 0$. We obtain $f(L_\alpha(p)) = \{x(t) + iy(t) : t \in \mathbb{R}\}$, where

$$x(t) = \sin(p_1 + \alpha t)\cosh(p_2 + t),$$

$$y(t) = \cos(p_1 + \alpha t)\sinh(p_2 + t).$$

Without loss of generality we may assume that $\alpha > 0$ (otherwise we can replace t by $-t$). Take an arbitrary $c \in (1, 2)$, consider the equation

$$x'_y(t) = \frac{\alpha + \tan(p_1 + \alpha t)\tanh(p_2 + t)}{-\alpha\tan(p_1 + \alpha t)\tanh(p_2 + t) + 1} = c\delta \tag{10}$$

for a fixed c.

Since $\tanh(p_2+t)\uparrow 1$ as $t\to+\infty$, for any $\varepsilon>0$ there exists $t_0(\varepsilon)$ such that the graph of the function

$$q(t)=\frac{|c\delta-\alpha|}{(1+c\delta\alpha)\tanh(p_2+t)},\ t\geq t_0(\varepsilon)$$

lies in the strip

$$\left\{(t,x):t\geq t_0(\varepsilon),\frac{|c\delta-\alpha|}{1+c\delta\alpha}\leq x\leq\frac{|c\delta-\alpha|(1+\varepsilon)}{1+c\delta\alpha}\right\}.$$

That is why for any fixed $c\in(1,2)$ equation (10) has infinitely many solutions $t_k(c)\to+\infty$ as $k\to\infty$, and there holds

$$\frac{|c\delta-\alpha|}{1+c\delta\alpha}\leq|\tan(p_1+\alpha t_k(c))|\leq\frac{|c\delta-\alpha|(1+\varepsilon)}{1+c\delta\alpha}.$$

Without loss of generality we may assume that $c\delta-\alpha>0$ for any $c\in(1,2)$. That is why we have the estimate

$$-p_1+\arctan\frac{|c\delta-\alpha|}{1+c\delta\alpha}+\pi k\leq t_k(c)\leq-p_1+\arctan\frac{|c\delta-\alpha|(1+\varepsilon)}{1+c\delta\alpha}+\pi k.$$

Getting two different values of c (say c_1 and c_2), we can choose $\varepsilon>0$ small enough for the following inequalities to hold for all big enough k (say for all $k\geq k_0$):

$$t_k(c_1)\leq-p_1+\arctan\frac{|c_1\delta-\alpha|(1+\varepsilon)}{1+c_1\delta\alpha}+\pi k$$

$$<-p_1+\arctan\frac{|c_2\delta-\alpha|}{1+c_2\delta\alpha}+\pi k\leq t_k(c_2).$$

The curve $\eta_k(t)=x(t)+iy(t)$, $t\in[t'_k,t''_k]$, has slope between $\frac{1}{2\delta}$ and $\frac{1}{\delta}$ for any $k\geq k_0$, where $t'_k=t_k(c_1)$, $t''_k=t_k(c_2)$.

Then

$$|\eta_k|=\int\limits_{t'_k}^{t''_k}\sqrt{(x'(t))^2+(y'(t))^2}dt$$

$$=\int\limits_{t'_k}^{t''_k}\sqrt{(\alpha^2+1)(\cosh^2(p_2+t)-\cos^2(p_1+\alpha t)-1)}dt\geq\frac{1}{4}\int\limits_{t'_k}^{t''_k}\exp(p_2+t)dt$$

$$\geq\frac{1}{4}\exp(p_2+t'_k)(t''_k-t'_k)\to\infty$$

as $k\to+\infty$. Hence, there exists $k\in\mathbb{N}$ such that $|\eta_k|\geq l$.

Remark 2. If $L_p(\alpha)$ is a non-oblique line, then we cannot state that the set $f^{\circ(3)}(L_\alpha(p))$ is dense in $\mathbb{C}$. Indeed, if we get $L_\alpha(p)$ as the imaginary axis $\mathrm{Re(z)} = 0$ and $f(z) = \mathrm{e}^z$, we obtain that the set $f^{\circ(3)}(L_\alpha(p)) \subset \mathbb{R}$ is contained in the ball $B(0, \exp(\exp(1)))$ and, hence, is not dense in $\mathbb{C}$.

Finally, we will study connection between density of iterations $f^{\circ(n)}(A)$ and $f^{\circ(m)}(A)$ of a set $A \subseteq \mathbb{C}$.

Theorem 3. *Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function such that the image of f is dense in $\mathbb{C}$. Let A be a subset of the complex plane. If there exists $n \in \mathbb{N}$ such that the set $f^{\circ(n)}(A)$ is dense in $\mathbb{C}$, then for any $\in \mathbb{N}$ the set $f^{\circ(n+m)}(A)$ is also dense in $\mathbb{C}$.*

Proof. It is sufficient to prove the theorem for $m = 1$. Let y_0 be an arbitrary point from the image of f, then there exists $x_0 \in \mathbb{C}$ such that $y_0 = f(x_0)$. Since the set $f^{\circ(n)}(A)$ is dense in $\mathbb{C}$, there exists a sequence $(x_n)_{k=1}^{\infty} \subset f^{\circ(n)}(A)$ such that $x_k \to x_0$ as $k \to \infty$. Due to continuity of f, we get $z_k = f(x_k) \to y_0$ as $k \to \infty$. It is clear that $z_k \in f^{\circ(n+1)}(A)$ for any $k \in \mathbb{N}$. Let r_0 be a point which does not belong to the image of f. There exists a sequence $(r_q)_{q=1}^{\infty}$ of points belonging to the image of f such that $r_q \to r_0$ as $q \to \infty$. As was shown, for any fixed $\varepsilon > 0$ for any $q \in \mathbb{N}$ one can find a point $z_q \in f^{\circ(n+m)}(A)$ such that $|z_q - r_q| \leq \varepsilon$. Therefore, there exists $\delta(\varepsilon) > 0$ such that for $q \geq \delta(\varepsilon)$ the inequalities hold

$$|z_q - r_0| \leq |z_q - r_q| + |r_q - r_0| \leq 2\varepsilon.$$

Hence, $z_q \to r_0$ as $q \to \infty$. The theorem is proved. $\square$

Remark 3. Theorem 3 implies that Theorem 1 is valid for all iterations of order at least 2 and Theorem 2 is valid for all iterations of order at least 3. Indeed, any function from class $\mathcal{A}$ or class $\mathcal{B}$ is continuous and has dense image.

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