ON HOLOMORPHIC AND PLURISUBHARMONIC FUNCTIONS

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Assume that Ω and Ω' are two bounded, connected, and open subsets of $\mathbb{C}$, and let $f : \Omega \to \Omega'$ be a function. Then it is a classical result that the following conditions are equivalent:

(1) *for any subharmonic function* $\varphi : \Omega' \to \mathbb{R} \cup \{-\infty\}$, *the function* $(\varphi \circ f) : \Omega \to \mathbb{R} \cup \{-\infty\}$ *is subharmonic.*

(2) *for any harmonic function* $h : \Omega' \to \mathbb{R}$, *the function* $(h \circ f) : \Omega \to \mathbb{R}$ *is harmonic;*

(3) f *is holomorphic or* $\bar{f}$ *is holomorphic;*

We shall prove the higher dimensional analog of the above theorem.

Theorem 1. *Let* $f = (f_1, \dots, f_M) : \Omega_N \to \Omega_M$ *be a function, where* $\Omega_N \subset \mathbb{C}^N$, *and* $\Omega_M \subset \mathbb{C}^M$, $N, M \geq 1$, *are bounded, connected, and open sets. Then the following assertions are equivalent:*

(1) *for any plurisubharmonic function* $\varphi : \Omega_M \to \mathbb{R} \cup \{-\infty\}$, *the function* $(\varphi \circ f) : \Omega_N \to \mathbb{R} \cup \{-\infty\}$ *is plurisubharmonic;*

(2) *for any pluriharmonic function* $h : \Omega_M \to \mathbb{R}$, *the function* $(h \circ f) : \Omega_N \to \mathbb{R}$ *is pluriharmonic;*

(3) f *is holomorphic or* $\bar{f}$ *is holomorphic.*

We shall also characterize those holomorphic functions which preserves the class of harmonic (subharmonic) functions.

Theorem 2. *Let* $f = (f_1, \dots, f_M) : \Omega_N \to \Omega_M$ *be a holomorphic mapping, where* $\Omega_N \subset \mathbb{C}^N$, $N \geq 1$, *and* $\Omega_M \subset \mathbb{C}^M$, $M > 1$, *are bounded, connected and open sets. Then the following assertions are equivalent:*

(1) *for any subharmonic function* $\varphi : \Omega_M \to \mathbb{R} \cup \{-\infty\}$, *the function* $(\varphi \circ f) : \Omega_N \to \mathbb{R} \cup \{-\infty\}$ *is subharmonic;*

(2) *for any harmonic function* $h : \Omega_M \to \mathbb{R}$, *the function* $(h \circ f) : \Omega_N \to \mathbb{R}$ *is harmonic;*

(3) f *has the following form:*

   (a) *if* $M \leq N$, *then* f *is constant or* f *can be represented by:*

$$f(z) = cAz + w_0 \,,$$

   where $c \in \mathbb{R}$, $w_0 \in \mathbb{C}^M$, *and* A *is a* $M \times N$*-unitary matrix;*

   (b) *if* $M > N$, *then* f *is constant.*