FAMILIES OF EXPOSING MAPS IN STRICTLY PSEUDOCONVEX DOMAINS

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Let $G \subset\subset \mathbb{C}^n$ be a domain and let $\zeta \in \partial G$. We say that ζ is a *globally strongly convex* boundary point of G if ∂G is of class $\mathcal{C}^2$ and strongly convex at ζ, and $\overline{G} \cap T_\zeta(\partial G) = \{\zeta\}$, where $T_\zeta(\partial G)$ denotes the tangent hyperplane of ∂G at ζ. It is known (cf. [2]) that

Theorem 0.1. *If G is strictly pseudoconvex and has boundary of class $\mathcal{C}^2$, then for every $\zeta \in \partial G$ there exist a neighbourhood $\hat{G}$ of $\overline{G}$ and a holomorphic embedding $h : \hat{G} \to \mathbb{C}^n$ such that $h(\zeta)$ is a globally strongly convex boundary point of $h(G)$.*

Such an h is called an *exposing mapping of G at ζ*. The following question has been formulated in [1]:

Problem 0.2. Let $\rho : \mathbb{D} \times \mathbb{C}^n \to \mathbb{R}$ be a plurisubharmonic function of class $\mathcal{C}^k, k \in \mathbb{N}, k \geq 2$. Assume that for any $t \in \mathbb{D}$ the truncated function $\rho|_{\{t\}\times\mathbb{C}^n}$ is strictly plurisubharmonic and globally defines a bounded strictly pseudoconvex domain $G_t := \{w \in \mathbb{C}^n : \rho(t, w) < 0\}$. This latter can be understood as a family of strictly pseudoconvex domains with boundaries of class $\mathcal{C}^k$ over $\mathbb{D}$. Do there exist $\mathcal{C}^{k-2}$-continuously varying family $(h_{t,\zeta})_{t\in\mathbb{D},\zeta\in\partial G_t}$ of exposing maps for G_t at $\zeta \in \partial G_t$?

We present the following (see [3])

Theorem 0.3. *Let $(G_t)_{t\in\mathbb{D}}$ be a family of strictly pseudoconvex domains as in Problem 0.2 with $k = 2$. Let $\sigma \in (0, 1)$. Take an $R > 0$ such that $\bigcup_{t\in\sigma\overline{\mathbb{D}}} \overline{G}_t \subset\subset \mathbb{B}(0, R)$. Assume that there exist a $\mathcal{C}^2$-continuous family $(\gamma_{t,\zeta})_{t\in\sigma\overline{\mathbb{D}},\zeta\in\partial G_t}$ of smooth embedded arcs $[0, 1] \to \mathbb{C}^n$ such that $\gamma_{t,\zeta}(0) = \zeta, \gamma_{t,\zeta}(1) \in \mathbb{S}^{2n-1}(R)$ and $\gamma_{t,\zeta}(x) \in \mathbb{C}^n \setminus (\overline{G_t} \cup \mathbb{S}^{2n-1}(R)), x \in (0, 1)$, for all $t \in \sigma\overline{\mathbb{D}}$ and $\zeta \in \partial G_t$. Then there exist a family $(h_{t,\zeta})_{t\in\sigma\overline{\mathbb{D}},\zeta\in\partial G_t}$ of exposing maps for G_t at ζ, continuous with respect to all variables.*

Here and below $\mathbb{B}(a, R)$ stands for the open ball in $\mathbb{C}^n$ with center at a and radius $R > 0$, and $\mathbb{S}^{2n-1}(R) := \partial\mathbb{B}(0, R)$.

Remark 0.4. Our assumption concerning the $\mathcal{C}^2$-continuity of the family $(\gamma_{t,\zeta})_{t\in\sigma\overline{\mathbb{D}},\zeta\in\partial G_t}$ should be understood in the following way:
For each t let Γ_t be a neighbourhood of ∂G_t with $\nabla r_t \neq 0$ on Γ_t, where $r_t := \rho(t, \cdot)$ and ∇r_t denotes its gradient. The neighbourhoods Γ_t may be chosen to depend in a $\mathcal{C}^2$-continuous way on t.

Then there exist positive constants $\sigma' \in (\sigma, 1)$ and $\tilde{\varepsilon}$ such that the family $(\gamma_{t,\zeta})_{t\in\sigma\overline{\mathbb{D}},\zeta\in\partial G_t}$ may be extended to a $\mathcal{C}^2$-continuous family

$$(\gamma_{t,\zeta})_{t\in\sigma'\mathbb{D},\zeta\in\bigcup_{|\kappa|<\tilde{\varepsilon}}\partial G_t^{(\kappa)}}$$

of smooth embedded arcs $[0,1] \to \mathbb{C}^n$ such that $\gamma_{t,\zeta}(0) = \zeta, \gamma_{t,\zeta}(1) \in \mathbb{S}^{2n-1}(R)$ and $\gamma_{t,\zeta}(x) \in \mathbb{C}^n \setminus (\overline{G_t^{(\kappa)}} \cup \mathbb{S}^{2n-1}(R)), x \in (0,1)$, for all $t \in \sigma'\mathbb{D}$ and $\zeta \in \partial G_t^{(\kappa)}, |\kappa|<\tilde{\varepsilon}$. Here, for small $|\kappa|$ we have put

$$G_t^{(\kappa)} := (G_t \setminus \Gamma_t) \cup \{z \in \Gamma_t : r_t(z) < \kappa\}.$$

References

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