On the geometry of the symmetrized bidisc

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We study the action of the automorphism group of the 2 complex dimensional manifold symmetrized bidisc $\mathbb{G}$ on itself. The automorphism group is 3 real dimensional. It foliates $\mathbb{G}$ into leaves all of which are 3 real dimensional hypersurfaces except one, viz., the royal variety. This leads us to investigate Isaev's classification of all Kobayashi-hyperbolic 2 complex dimensional manifolds for which the group of holomorphic automorphisms has real dimension 3 studied by Isaev. Indeed, we produce a biholomorphism between the symmetrized bidisc and the domain

$$\left\{(z_1, z_2) \in \mathbb{C}^2 : 1 + |z_1|^2 - |z_2|^2 > |1 + z_1^2 - z_2^2|,\ \mathrm{Im}(z_1(1 + \overline{z_2})) > 0\right\}$$

in Isaev's list. Isaev calls it $\mathcal{D}_1$. The road to the biholomorphism is paved with various geometric insights about $\mathbb{G}$. Several consequences of the biholomorphism follow including two new characterizations of the symmetrized bidisc and several new characterizations of $\mathcal{D}_1$. Among the results on $\mathcal{D}_1$, of particular interest is the fact that $\mathcal{D}_1$ is a "symmetrization". When we symmetrize (appropriately defined in the context) either Ω_1 or $\mathcal{D}_1^{(2)}$ (Isaev's notation), we get $\mathcal{D}_1$. These two domains Ω_1 and $\mathcal{D}_1^{(2)}$ are in Isaev's list and he mentioned that these are biholomorphic to $\mathbb{D} \times \mathbb{D}$. We produce explicit biholomorphisms between these domains and $\mathbb{D} \times \mathbb{D}$.