ON A FAMILY OF QUASIMETRIC SPACES IN GENERALIZED POTENTIAL THEORY

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Let X be a non-empty set, and let $d : X \times X \to [0, \infty)$ be a function that satisfies:

(1) $d(x, y) = 0$ if, and only if, $x = y$;
(2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(3) there exists a constant $C \geq 1$ such that

$$d(x, y) \leq C(d(x, z) + d(z, y))$$

for all $x, y, z \in X$.

We shall call d for a *quasimetric*, and the pair (X, d) for a *quasimetric space*.

Let $n \geq 2$, $1 \leq m \leq n$, and assume that $\Omega \subset \mathbb{C}^n$ is a m-hyperconvex domain. For $u, v \in \mathcal{E}_{p,m}(\Omega)$ and $p > 0$ let us define

$$\mathcal{J}_p(u, v) = \left(\int_\Omega |u - v|^p (\mathrm{H}_m(u) + \mathrm{H}_m(v)) \right)^{\frac{1}{p+m}}.$$

We shall prove that the set of m-subharmonic function with function $\mathcal{J}_p$, i.e. the space $(\mathcal{SH}_M(\Omega), \mathcal{J}_p)$ is the quasimetric space.

We shall also prove the following convergence results in the space $(\mathcal{E}_{p,m}(\Omega), \mathcal{J}_p)$:

(1) If $\mathcal{J}_p(u_j, u) \to 0$, as $j \to \infty$, then $u_j \to u$ in $L^{p+m}(\Omega)$.
(2) If $\mathcal{J}_p(u_j, u) \to 0$, as $j \to \infty$, then $u_j \to u$ in capacity cap_m.
(3) The inverse implications of the above results are not, in general valid.
(4) If $\mathcal{J}_p(u_j, u) \to 0$, as $j \to \infty$, then $\mathrm{H}_m(u_j) \to \mathrm{H}_m(u)$ weakly.

Let

$$\mathcal{M}_p = \big\{\mu \,:\, \mu \text{ is a non-negative Radon measure on } \Omega \text{ such that } \mathrm{H}_m(u) = \mu \text{ for some } u \in \mathcal{E}_{p,m}\big\}.$$

The following result concerning stability is true for m-subharmonic functions. Let $n \geq 2$, $1 \leq m \leq n$, and assume that $\Omega \subset \mathbb{C}^n$ is a m-hyperconvex domain and let $\mu \in \mathcal{M}_p$. If $0 \leq f, f_j \leq 1$ are measurable functions such that $f_j \to f$ in $L^1_{loc}(\mu)$, as $j \to +\infty$, then $\mathcal{J}_p(U(f_j\mu), U(f\mu)) \to 0$, where $U(\mu)$ is the unique solution to the following Dirichlet problem: $\mathrm{H}_m(U(\mu)) = \mu$.

Finally we are going to show that the corresponding result holds also in the case of compact Kähler manifold. Let $n \geq 2$, $p > 0$, and let $1 \leq m \leq n$. Assume that (X, ω) is a connected and compact Kähler manifold of complex dimension n, where ω is a Kähler form on X such that $\int_X \omega^n = 1$. For $u, v \in \mathcal{E}_{p,m}(X, \omega)$ and we define

$$\mathrm{I}_p(u, v) = \left(\int_X |u - v|^p (\mathrm{H}_m(u) + \mathrm{H}_m(v)) \right)^{\frac{1}{p+m}}.$$

We proved that the pair $(\mathcal{E}_{p,m}(X, \omega), \mathrm{I}_p)$ is a complete quasimetric space.