ON THE DIRICHLET PROBLEM FOR THE COMPLEX MONGE-AMPÈRE OPERATOR

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Let us recall the following Cegrell classes of plurisubharmonic functions defined in bounded hyperconvex domain $\Omega\subset\mathbb{C}^n$:

$$\mathcal{E}_0=\left\{\varphi\in\mathcal{PSH}\cap L^\infty:\forall\xi\in\partial\Omega\lim_{\Omega\ni z\to\xi}\varphi(z)=0,\int_\Omega(dd^c\varphi)^n<\infty\right\}.$$

$$\mathcal{E}_p=\left\{u\in\mathcal{PSH}:\exists\{u_j\}_{j=1}^\infty\subset\mathcal{E}_0,u_j\searrow u,\sup_j\int_\Omega(-u_j)^p(dd^cu_j)^n<\infty\right\}.$$

$$\mathcal{F}=\left\{u\in\mathcal{PSH}^-(\Omega):\exists\{u_j\}_{j=1}^\infty\subset\mathcal{E}_0,u_j\searrow u,\sup_j\int_\Omega(dd^cu_j)^n<+\infty\right\}.$$

$$\mathcal{E}=\left\{u\in\mathcal{PSH}^-(\Omega):\forall z\in\Omega\,\exists\omega_z\subset\Omega\,\exists u_z\in\mathcal{F},u=u_z\text{ na }\omega_z\right\}.$$

$$\mathcal{N}=\{u\in\mathcal{E}:\tilde{u}=0\},$$

where $\tilde{u}$ is the smalest maximal plurisubharmonic majorant of u. Let $\mathcal{K}\in\{\,\mathcal{E}_0,\,\mathcal{E}_p,\mathcal{F},\,\mathcal{N},\mathcal{E}\}$. Then $u\in\mathcal{K}(H)$, $H\in\mathcal{E}$, if there exists $\varphi\in\mathcal{K}$ such that

$$H\geq u\geq\varphi+H\,.$$

By the Dirichlet problem for the complex Monge-Ampère operator in the class $\mathcal{K}(H)$ we mean: for a positive measure μ in Ω, and $H\in\mathcal{E}$ find a plurisubharmonic function $u\in\mathcal{K}(H)$ such that

$$(dd^cu)^n=\mu.$$

Let $\mathcal{M}_{MA}=(dd^c\cdot)^n(\mathcal{E})$ and $\mathcal{M}_p=(dd^c\cdot)^n(\mathcal{E}_p)$.

Cegrell proved that for any measure μ in hyperconvex domain we get the following decomposition:

$$\mu=f\,(dd^c\phi)^n+\nu=\mu_r+\mu_s,$$

where $\phi\in\mathcal{E}_0$, $f\in L^1_{loc}((dd^c\phi)^n)$, $f\geq 0$ and μ_s is positive measure supported by pluripolar set.

(1) If $\mu\in\mathcal{M}_{MA}$, then $t\mu\in\mathcal{M}_{MA}$.
(2) If $\mu_1,\mu_2\in\mathcal{M}_{MA}$, then $\mu_1+\mu_2\in\mathcal{M}_{MA}$.
(3) If $\mu\in\mathcal{M}_{MA}$ and $\nu\leq\mu$, then $\nu\in\mathcal{M}_{MA}$.
(4) If $\mu=(dd^cu)^n=\mu_r(u)+\mu_s(u)$, then $\mu_r(u)=fdd^c(dd^c\psi)^n$, for some $\psi\in\mathcal{E}_0$, $f\geq 0$, $f\in L^1_{loc}((dd^c\psi)^n)$ and $supp\mu_s(u)\subset\{u=-\infty\}$.
(5) It follows from subsolution theorem that in the Dirichlet problem it is enough to consider regular and singular measures separately.

If $u_1 \in \mathcal{E}(\Omega_1)$, $u_2 \in \mathcal{E}(\Omega_2)$ are such that $\mu_s(u_1) = \chi_{\{u_1=-\infty\}}(dd^c u_1)^{n_1}$ and $\mu_s(u_2) = \chi_{\{u_2=-\infty\}}(dd^c u)^{n_2}$, then $\max(u_1, u_2) \in \mathcal{E}(\Omega_1 \times \Omega_2)$ and

$$\mu_s(\max(u_1, u_2)) = \mu_s(u_1) \otimes \mu_s(u_2).$$

Open question: Can we find plurisubharmonic function u such that

$$(dd^c u)^n = \mu_r(u_1) \otimes \mu_s(u_2).$$

Even for

$$\mu = dV \otimes \delta_0$$

the answer is unknown. We know that there is no toric symmetric solution $u(z, w) = u(|z|, w)$ or $u(z, w) = u(z, |w|)$.

Open question: is it true, as in the singular case, that for $u_1 \in \mathcal{E}(\Omega_1)$, $u_2 \in \mathcal{E}(\Omega_2)$

$$\mu_r(u_1) \otimes \mu_r(u_2) \in \mathcal{M}_{MA}?$$

In general the answer is unknown. In some special cases: the answer is YES!

(1) If measures $\mu_r(u_1)$, $\mu_r(u_2)$ are finite then the answer is YES, since the product measure would be finite, and the solution would be in the class $\mathcal{F}$.
(2) If u_1 and u_2 are bounded functions, then $u_1, u_2 \in \mathcal{PSH}(\Omega_1 \times \Omega_2)$. In this case Monge-Ampère measures can be infinite!
(3) If μ_1 is finite, μ_2 is such that there exists negative plurisubharmonic function ψ in Ω_2 such that $\psi \in L^1(\mu_2)$, then the answer is YES!
(4) If $\mu_1 \in \mathcal{M}_{p_1}$, $\mu_2 \in \mathcal{M}_{p_2}$ then $\mu_1 \otimes \mu_2 \in \mathcal{M}_q$, for $q < \frac{\alpha(n_1+n_2)}{1-\alpha}$.
(5) If $u_1, u_2 \in \mathcal{E}(\Omega_1 \times \Omega_2)$ then the answer is YES!

Let μ be a unitary invariant measure defined on $B(0, 1)$, and let $F(t) = \frac{1}{(2\pi)^n}\mu(B(0, t))$. TFCAE:

(1) $\mu \in \mathcal{M}_{MA}$,
(2) $\int_{\frac{1}{2}}^{1} \sqrt[n]{F(t)}\, dt < \infty$.

It follows from the following representation

$$u(z) = \int_{|z|}^{1} -\frac{1}{t} \sqrt[n]{F(t)}\, dt.$$

Open question: Assume that the density function $f \in L^p(B(0, 1))$, $p > 0$ and $\mu = f dV$.

- If $p > 1$ then the solution is continuous (Hölder continuous).
- If $p = 1$ the solution is in the class $\mathcal{F}$.
- If $p < \frac{1}{n+1}$ then $f(z) = (1 - |z|^{2n})^{-n-1} \in L^p$, but

$$F(t) = C((1 - t^{2n})^{-n} - 1) \notin L^{\frac{1}{n}}([\frac{1}{2}, 1]).$$

  Therefore there is no solution to the Dirichlet problem.
- What for $p \in [\frac{1}{n+1}, 1)$?