Tao's result on Sendov's conjecture
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This talk, spanning over three meetings (13.05.2024, 20.05.2024, 27.05.2024), is the ultimate one in the joint effort to understand the paper of Terence Tao on the following conjecture of Sendov:

Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = a_n \prod_{j=1}^{n} (z - z_j)$$

be a complex polynomial ($P \in \mathcal{P}$) of degree $n \geq 2$ with all its zeros in the closed unit disc $\overline{\mathbb{D}}$ (i.e., $z_j \in \overline{\mathbb{D}}$, $j = 1, \ldots, n$).

Conjecture of Sendov'1958: *For each $j \in \{1, \ldots, n\}$, the closed disc $\overline{\mathbb{D}}(z_j; 1)$ contains a critical point of P.*

The result of Terence Tao is: *There exists $n_0 \in \mathbb{N}$ such that for all polynomials of degree $n > n_0$, satisfying the assumptions, Sendov's conjecture holds.*

An explicit bound on n_0 is not known, and even if it was possible to refine the arguments of Tao to yield a precise value of n_0, it would be far too large to give conclusive result for the full Sendov's conjecture, which remains open.

We present:

The Reduction of the general problem: It is enough to consider the situation where f is a monic polynomial and the zero z_j (for fixed j) is real and non-negative ($z_j = a \in [0, 1]$).

The main result [Theorem 1.3 in the paper]. *Let n range over a sequence of natural numbers going to infinity. For each n in this sequence, let $f = f^{(n)}$ be a monic polynomial of degree n with all zeros in $\overline{D(0,1)}$, and let $a = a^{(n)} \in [0, 1]$ be such that $f(a) = 0$. Suppose also that, for every n in the sequence, f' has no zeros in $\overline{D(a, 1)}$ (so, $f^{(n)} \in \mathcal{P}$ is a sequence of counterexamples). Then, one can derive a contradiction.*

We get a sequence $a^{(n)} \in [0, 1]$. After passing to a subsequence we may WLOG assume that $a^{(n)} \to a^{(\infty)} \in [0, 1]$. There are three cases of qualitatively different natures:

- $a^{(\infty)} \in (0, 1)$
- $a^{(\infty)} = 0$
- $a^{(\infty)} = 1$

(In increasing order of difficulty)

On the 13th of May we presented:

- Introduction of the notions.
- Some discussion of general nature on the result.
- The "toolbox" ([Lemma 1.6 in the paper]) on the relations between the Logarithmic potentials, Cauchy transforms and the distribution of zeros of the polynomials in the "would be" counterexample sequence and their derivatives.
- A lemma [Theorem 1.10 in the paper] on the description of the limiting measures (weak limits of convex combinations of Dirac deltas, concentrated at the roots of f and f').

On the 20th of May we presented:

- A theorem [Theorem 3.1 in the paper], which covers the case when $a^{(\infty)} = 0$.
- A technical lemma [Lemma 5.2 in the paper] on the preliminary bounds of f and f'.
- A lemma [Proposition 5.3 in the paper] on the description of the cluster set of the zeros of (a subsequence of) the derivatives of f.
- A lemma [Proposition 5.4 in the paper] on the approximation of f and f' outside of the so defined cluster set.

On the 27th of May we presented:

- A technical lemma [Corollary 5.5 in the paper], which says that the zeros of f lie very close to a level set of the logarithmic potential of ζ.
- A long and technical lemma [Proposition 5.6 in the paper] on the fine control on ζ.
- The final proof of Theorem 1.3 in the case $a^{(\infty)} = 1$.

The presentation follows very closely the exposition in T.Tao, *Sendov's conjecture for sufficiently-high-degree polynomials*, Acta Math., **229** (2022), 347 – 392, DOI:10.4310/ACTA.2022.v229.n2.a3
with an effort to explain all the technicalities.