Complex Analysis and geometric function theory seminar, Jagiellonian University

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Composition Operators and Rational Inner Functions on the bidisc.

Abstract:
Given a Banach space X, a domain $\Omega \subset \mathbb{C}^n$ and a holomorphic function f on X, the composition operator is defined as $C_\Phi(f) = f \circ \Phi$ for $\Phi : \Omega \to \Omega$ a holomorphic self-map of Ω. In the past years, the study of the composition operators has attracted a lot of attention, after the seminal work of J.Shapiro for Hardy and Bergman spaces on the unit disc $\mathbb{D}$. More recently, the focus has turned to other domains and settings, in general. One of the settings that the study of the boundedness, compactness and other properties of the composition operator is relatively new, is in domains of $\mathbb{C}^n$. Several authors have provided results for the Hardy and Bergman space on the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, or the polydisc $\mathbb{D}^n$. To be precise, Woken gave a characterisation of the holomorphic self-maps of the unit ball (which are smooth up to the boundary) that induce bounded composition operators is given, in terms of the first derivative of the symbol Φ. In the works of Bayart and Kosiński, a full characterization of the holomorphic self-maps of the bidisc (again requiring some smoothness up to the distinguished boundary $\mathbb{T}^2$) is provided. Both of these works are quite elegant and technical but only take care of the case where the symbol $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$ has coordinate functions that have some type of smoothness on the distinguished boundary of the bidisc $\mathbb{D}^2$. In the present talk, the case where the symbol has some type of singularity on the boundary is considered. Some interesting results arise when we consider the coordinate functions to be Rational Functions and Rational Inner Functions defined on the bidisc.
Rational Inner Functions and their properties are currently studied, and several authors have obtained a substantial amount of very interesting results. In the works of Sola, Knese, Bickel, Pascoe, M^{c}Carthy, Bergqvist and others (see [**?**], [**?**], [**?**], [**?**], [**?**], [**?**]) one can find results about the regularity and smoothness of Rational Inner Functions (or RIFs) and membership of RIFs in holomoprhic function spaces (Hardy, Dirichlet spaces).
In the present work some of these results will be applied to prove some new theorems regarding the boundedness of the composition operator when the symbol Φ has RIFs as coordinate functions, combining them with the results of Kosiński

and Bayart in [**?**], [**?**]. The main goal of this note is to answer the following question.

Question 1: *Assume that* $\Phi : \mathbb{D}^2 \to \mathbb{D}^2$, *where* $\Phi = (\varphi_1, \varphi_2)$ *is a holomorphic self map of the bidisc and at least one of the coordinate functions* $\varphi_i, i = 1, 2$ *is a Rational Inner Function on the bidisc, hence not always smooth on the distinguished boundary* $\mathbb{T}^2$. *What can one say about the boundedness of the composition operator* $C_\Phi : A^2_\beta(\mathbb{D}^2) \to A^2_\beta(\mathbb{D}^2)$ *induced by such symbols?*

A secondary target of this talk is to study the boundedness of the composition operator C_Φ acting on the anisotropic Dirichlet space $\mathfrak{D}_{\vec{a}}(\mathbb{D}^2)$, where $\vec{a} = (a_1, a_2) \in \mathbb{R}^2$.