DISCONTINUITY OF THE LEMPERT FUNCTION OF THE SPECTRAL BALL

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1. Introduction and statement of results

The *spectral ball* Ω_n is the set of all $n \times n$ complex matrices with eigenvalues strictly smaller than one in modulus. It can be seen as the union of all the unit balls of the space of matrices endowed with all the possible operator norms arising from a choice of norm on the space $\mathbb{C}^n$. It is unbounded and very far from being hyperbolic — in particular it contains many entire curves. As analogues of the Montel theorem cannot hold for mappings with values in the spectral ball, several classical invariant objects in complex analysis exhibit discontinuity phenomena in this setting.

The Lempert function of a domain $D \subset \mathbb{C}^m$ is defined, for $z, w \in D$, as

$$l_D(z,w) := \inf\{|\alpha| : \alpha \in \mathbb{D} \text{ and } \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \varphi(\alpha) = w\}.$$

The Lempert function is always upper semicontinuous.

For $A \in \mathcal{M}_n$ denote by $sp(A)$ and $r(A) = \max_{\lambda \in sp(A)} |\lambda|$ the spectrum and the spectral radius of A, respectively. The characteristic polynomial of the matrix A is

$$P_A(t) := \det(tI - A) =: t^n + \sum_{j=1}^{n} (-1)^j \sigma_j(A) t^{n-j},$$

where $I \in \mathcal{M}_n$ is the unit matrix. We define a map σ from $\mathcal{M}_n$ to $\mathbb{C}^n$ by $\sigma := (\sigma_1, \dots, \sigma_n)$. The *symmetrized polydisk* is $\mathbb{G}_n := \sigma(\Omega_n)$ is a bounded domain in $\mathbb{C}^n$, which is hyperconvex, and Kobayashi complete hyperbolic.

The Lempert function decreases under holomorphic maps, so

$$l_{\Omega_n}(A, B) \geq l_{\mathbb{G}_n}(\sigma(A), \sigma(B)). \tag{1.1}$$

A matrix A is *cyclic* (or *non-derogatory*) if it admits a cyclic vector. We denote by $C_{\sigma(A)}$ the companion matrix of the characteristic polynomial of A; A is cyclic if and only if it is conjugate to $C_{\sigma(A)}$.

1991 *Mathematics Subject Classification.* [2000] 32U35, 32F45.

Proposition 1.1 (Agler-Young)**.** *If $A, B \in \Omega_n$ are cyclic, then*

$$l_{\Omega_n}(A,B) = l_{\mathbb{G}_n}(\sigma(A),\sigma(B)). \tag{1.2}$$

A bit more generally:

Proposition 1.2. *Let $A, B \in \Omega_n$.*

(1) *The Lempert function l_{Ω_n} is continuous at (A,B) if and only if* (1.2) *holds.*
(2) *If B is cyclic, and the function $l_{\Omega_n}(.,B)$ is continuous at A, then* (1.2) *holds.*

Our main result looks at the arguments of the Lempert function separately:

Theorem 1.3. *Let $A \in \Omega_n$. Then A is cyclic if and only if the function $l_{\Omega_n}(.,B)$ is continuous at A for all $B \in \Omega_n$.*

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