NOTE ON THE GROUP OF AUTOMORPHISMS OF THE SPECTRAL BALL

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ABSTRACT. It is shown that the conjecture on the description of the group of automorphism of the spectral ball posed by Ransford and White is false.

Let Ω denote the spectral ball, that is a domain composed of $n \times n$ complex matrices whose spectral radius is less then 1. The natural question that arises in the study of the geometry of Ω is to classify its group of automorphisms. It is seen that among them there are the following three forms:

(i) *Transposition*: $\mathcal{T} : x \mapsto x^t$,
(ii) *Möbius maps*: $\mathcal{M}_{\alpha,\gamma} : x \mapsto \gamma(x - \alpha)(1 - \overline{\alpha}x)^{-1}$, where α lies in the unit disc and $|\gamma| = 1$,
(iii) *Conjugations*: $\mathcal{J}_u : x \mapsto u(x)^{-1}xu(x)$, where $u : \Omega \to M_n^{-1}$ is a conjugate invariant holomorphic map, i.e. $u(q^{-1}xq) = u(x)$ for each $x \in \Omega$ and $q \in \mathcal{M}_n^{-1}$.

Ransford and White have asked in [3] whether the compositions of the three above forms generate the whole $\mathrm{Aut}(\Omega)$ - the group of automorphisms of the spectral ball. The problems related to this question were considered among others in [1], [2], [4] and [5].

The aim of this short note is to show that the answer to the conjecture posed by Ransford and White is negative. More precisely, we will present an example of an automorphism of the spectral ball which is not generated by mappings (i), (ii) and (iii).

To simplify the notation we focus our attention on the case $n = 2$. The similar examples working for $n \geq 2$ are self-evident.

For the convenience of the reader we start with recalling some basic properties of the mappings (i), (ii) and (iii) which will be useful in the sequel. Let f be an automorphism of the spectral ball generated by (i), (ii) and (iii) and preserving the origin. Observe that $\mathcal{J}_u \circ \mathcal{M}_{\alpha,\gamma} = \mathcal{M}_{\alpha,\gamma} \circ \mathcal{J}_{u \circ \mathcal{M}_{\alpha,\gamma}}$ for any $u : \Omega \to \mathcal{M}_n^{-1}$, $|\alpha| < 1$ and $|\gamma| = 1$. Therefore, the automorphism f is generated only by the mappings of the form (i) and (iii). Moreover, $f'(0)$ is a linear automorphism of the spectral ball and it follows from [3] that $f'(0).x = \gamma m x m^{-1}$, $x \in \Omega$, or $f'(0).x = \gamma m x^t m^{-1}$, $x \in \Omega$, for some

unimodular constant γ and invertible matrix m. Note also that if the first possibility holds, i.e. $f'(0).x = \gamma m x m^{-1}$, then $\gamma^{-1} f$ is of the form (iii).

Theorem 1 (Example 1). *Let φ be an entire holomorphic function on $\mathbb{C}$. Put*

$$f((x_{ij})) = \begin{pmatrix} x_{11} & e^{-\varphi(x_{12}x_{21})}x_{12} \\ e^{\varphi(x_{12}x_{21})}x_{21} & x_{22} \end{pmatrix}, \quad x \in \Omega.$$

Then f is an automorphism of Ω which is not generated by mappings of the form (i), (ii) and (iii), providing that φ is non-constant.

Proof. It is clear that f preserves the spectrum, so $f(\Omega) \subset \Omega$. Thus, it suffices to observe that the inverse to f is given by the formula:

$$f^{(-1)}((x_{ij})) = \begin{pmatrix} x_{11} & e^{\varphi(x_{12}x_{21})}x_{12} \\ e^{-\varphi(x_{12}x_{21})}x_{21} & x_{22} \end{pmatrix}, \quad x \in \Omega.$$

What remains to do is to show that f is not generated by the mappings of the form (i), (ii) and (iii).

It is clear that $f'(0).x = mxm^{-1}$, $x \in \Omega$, for some diagonal $m \in \mathcal{M}_2^{-1}$, so we aim at proving that f is not a conjugation $\mathcal{J}_v$ with v depending only on the conjugacy class of x.

It is enough to observe that f restricted to the fiber $\mathfrak{S} := \{x \in \Omega : \det x = 0, \ \operatorname{tr} x = 1/2\}$ is not of the form $f(x) = nxn^{-1}$, $x \in \Omega$, for any invertible n. To do it compute

$$f\left(\begin{pmatrix} \lambda & \lambda \\ 1/2 - \lambda & 1/2 - \lambda \end{pmatrix}\right) = \begin{pmatrix} \lambda & e^{-\varphi(\lambda(1/2-\lambda))}\lambda \\ e^{\varphi(\lambda(1/2-\lambda))}(1/2 - \lambda) & 1/2 - \lambda, \end{pmatrix},$$

$\lambda \in \mathbb{C}$, and use the trivial fact that for any $\alpha, \beta \in \mathbb{C}$ the mappings $e^{\varphi(\lambda(1/2-\lambda))}\lambda$ and $\alpha\lambda + \beta$, $\lambda \in \mathbb{C}$ are not identically equal, provided that φ is non-constant (compare for example the singularity at ∞). $\square$

Remark 1. One can show that the conjugation

$$(1) \qquad \mathcal{J}(x) = \operatorname{diag}(a(x), a^{-1}(x))x\operatorname{diag}(a(x), a^{-1}(x))^{-1},$$

where a is a non-vanishing holomorphic function on Ω, is an automorphism of the spectral ball if and only if a depends only on x_{11}, x_{22} and $x_{12}x_{21}$.

This remark describes more precisely the situation occurring in the Example 1. Actually, the automorphism appearing in Example 1 is of the form (1) with a depending only on $x_{12}x_{21}$ (more precisely, $a(x) = e^{-\varphi(x_{12}x_{21})}$).

Example 2. Put $u\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) := \begin{pmatrix} 1 & 0 \\ a(x_{12}) & 1 \end{pmatrix}$, $x \in \Omega$, where a is an entire holomorphic function. Then the conjugation $f(x) = u(x)xu(x)^{-1}$, $x \in \Omega$, is an automorphism of Ω non-generated by the automorphisms of the form (i), (ii) and (iii).

Proof. Direct computations allow us to observe that $f^{(-1)}$ is given by the formula $f^{(-1)}(x) = u^{-1}(x)xu(x)$, $x \in \Omega$. The second part may be proved similarly as in the previous example. $\square$

Remark 2. We would like to point out that the situation occurring in the above example may be described more precisely. Namely, put $u :=$ $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, where a is an arbitrary holomorphic function on Ω. Then the conjugation $\mathcal{J}_u : x \mapsto u(x)xu(x)^{-1}$ is an automorphism of Ω if and only if a depends on x_{12}, $\operatorname{tr} x$ and $\det x$.

The problem of describing the group of automorphisms of spectral ball was divided by Rostand ([4]) into two parts. The first one is to prove that any normalized automorphism (i.e. an automorphism fixing the origin and satysfying $f'(0) = \operatorname{id}$) is a conjugation $\mathcal{J}_u$, where $u : \Omega \to \mathcal{M}_n^{-1}$ is a holomorphic mapping. The second one is to show that the automorphism being a conjugation is of the form $\mathcal{J}_u$ with u satisfying the condition $u(q^{-1}xq) = u(x)$ for each $x \in \Omega$ and each invertible q. Note that it was shown in this paper that the second statement is false. On the other hand it seems that the answer to the first question is affirmative. So we have two following problems:

- Is every normalized automorphism of the spectral ball of the form $x \to u(x)xu(x)^{-1}$, where $u : \Omega \to \mathcal{M}_n^{-1}$ is a holomorphic mapping?
- Describe all holomorphic $u : \Omega \to \mathcal{M}_n^{-1}$ such that the mapping $x \mapsto u(x)xu(x)^{-1}$ is an automorphism of Ω.

References

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