An extension theorem with analytic singularities for generalized $(\mathbf{N},\mathbf{k})-$crosses.

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Abstract.

Let D_j be a Riemann domain over $\mathbb{C}^{n_j}$ and let $\varnothing \neq A_j \subset D_j$ for $j = 1, \dots, N$. For $k \in \{1, \dots, N\}$ let $I(N,k) := \{\alpha = (\alpha_1, \dots, \alpha_N) \in \{0,1\}^N : |\alpha| = k\}$, where $|\alpha| := \alpha_1 + \ldots + \alpha_N$. Put

$$\mathcal{X}_{\alpha,j} := \begin{cases} D_j, & \text{if } \alpha_j = 1 \\ A_j, & \text{if } \alpha_j = 0 \end{cases}, \quad \mathcal{X}_\alpha := \prod_{j=1}^N \mathcal{X}_{\alpha,j}.$$

For $\alpha \in I(N,k)$ such that $\alpha_{j_1} = \ldots = \alpha_{j_k} = 1, \alpha_{i_1} = \ldots = \alpha_{i_{N-k}} = 0$, where $j_1 < \ldots < j_k$ and $i_1 < \ldots < i_{N-k}$, put

$$D_\alpha := \prod_{s=1}^k D_{\alpha_{j_s}}, \quad A_\alpha := \prod_{s=1}^{N-k} A_{\alpha_{i_s}}.$$

For $a = (a_1, \dots, a_n) \in \mathcal{X}_\alpha$, where α is as above, put $a_\alpha^0 := (a_{\alpha_{i_1}}, \dots, a_{\alpha_{i_{N-k}}}) \in A_\alpha$. Analogously, define $a_\alpha^1 := (a_{\alpha_{j_1}}, \dots, a_{\alpha_{j_k}}) \in D_\alpha$. For any $a = (a_1, \dots, a_N) \in \prod_{j=1}^N A_j$ and $\alpha \in I(N,k)$ define

$$\boldsymbol{i}_{a,\alpha} = (\boldsymbol{i}_{a,\alpha,1}, \dots, \boldsymbol{i}_{a,\alpha,N}) : D_\alpha \to \mathcal{X}_\alpha, \quad \boldsymbol{i}_{a,\alpha,j}(z) := \begin{cases} z_j, & \text{if } \alpha_j = 1 \\ a_j, & \text{if } \alpha_j = 0 \end{cases}, \quad j = 1, \dots, N.$$

Similarly, for any $b = (b_1, \dots, b_N) \in \prod_{j=1}^N D_j$ and $\alpha \in I(N,k)$ define

$$\boldsymbol{l}_{b,\alpha} = (\boldsymbol{l}_{b,\alpha,1}, \dots, \boldsymbol{l}_{b,\alpha,N}) : A_\alpha \to \mathcal{X}_\alpha, \quad \boldsymbol{l}_{b,\alpha,j}(z) := \begin{cases} z_j, & \text{if } \alpha_j = 0 \\ b_j, & \text{if } \alpha_j = 1 \end{cases}, \quad j = 1, \dots, N.$$

Definition 0.1. For any $\alpha \in I(N,k)$ let $\Sigma_\alpha \subset A_\alpha$. We define a *generalized* $(N,k)-$*cross*

$$\mathbf{T}_{N,k} := \mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha \in I(N,k)}) = \bigcup_{\alpha \in I(N,k)} \{a \in \mathcal{X}_\alpha : a_\alpha^0 \notin \Sigma_\alpha\}$$

and its *center*

$$\mathfrak{C}(\mathbf{T}_{N,k}) := \mathbf{T}_{N,k} \cap (A_1 \times \ldots \times A_N).$$

Definition 0.2. Let

$$\widehat{\mathbf{X}}_{N,k} = \widehat{\mathbb{X}}_{N,k}((A_j, D_j)_{j=1}^N) := \{(z_1, \dots, z_N) \in D_1 \times \ldots D_N : \sum_{j=1}^N h^\star_{A_j,D_j}(z_j) < k\}.$$

Definition 0.3. For a relatively closed set $M \subset \mathbf{T}_{N,k}$[1] we say that a function $f : \mathbf{T}_{N,k} \setminus M \to \mathbb{C}$ is *separately holomorphic on* $\mathbf{T}_{N,k} \setminus M$ if for every $\alpha \in I(N,k)$ and for every $a \in \prod_{j=1}^N A_j$ with $a_\alpha^0 \in A_\alpha \setminus \Sigma_\alpha$ the function

$$D_\alpha \setminus M_{a_\alpha^0} \ni z \mapsto f(\boldsymbol{i}_{a,\alpha}(z))$$

is holomorphic, where $M_{a_\alpha^0} := \{z \in D_\alpha : \boldsymbol{i}_{a_\alpha^0,\alpha}(z) \in M\}$. In this case we write $f \in \mathcal{O}_s(\mathbf{T}_{N,k} \setminus M)$.

[1] We allow $M = \varnothing$ here.

We denote by $\mathcal{O}_s^c(\mathbf{T}_{N,k} \setminus M)$ the space of all $f \in \mathcal{O}_s(\mathbf{T}_{N,k} \setminus M)$ such that for any $\alpha \in I(N,k)$ and for every $b \in \prod_{j=1}^{N} D_j$ the function

$$A_\alpha \setminus (\Sigma_\alpha \cup M_{b_\alpha^1}) \ni z \mapsto f(\boldsymbol{l}_{b,\alpha}(z))$$

is continuous, where $M_{b_\alpha^1} := \{z \in A_\alpha : \boldsymbol{l}_{b_\alpha^1,\alpha}(z) \in M\}$.

We will prove the following

Theorem 0.4 (Extension theorem for generalized $(N,k)-$crosses with analytic singularities)**.** *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ be locally pluriregular, $j = 1, \ldots, N$, and let $\Sigma_\alpha \subset A_\alpha$ be pluripolar, $\alpha \in I(N,k)$. Let $\mathbf{X}_{N,k} := \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N), \mathbf{T}_{N,k} := \mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha \in I(N,k)})$. Define $M := \mathbf{T}_{N,k} \cap S$, where S is an analytic subset of $\widehat{\mathbf{X}}_{N,k}$ with* $\operatorname{codim} S \geq 1$ *and let $\mathcal{F} := \mathcal{O}_s^c(\mathbf{T}_{N,k} \setminus M)$. Denote by $\widehat{M}$ the union of all irreducible components of S of codimension one. Then:*

- *for any $f \in \mathcal{F}$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{M})$ such that $\widehat{f} = f$ on $\mathbf{T}_{N,k} \setminus M$,*
- *$\widehat{M}$ is singular with respect to the family $\{\widehat{f} : f \in \mathcal{F}\}$,*
- *$\widehat{f}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{M}) \subset f(\mathbf{T}_{N,k} \setminus M)$ for any $f \in \mathcal{F}$.*

Moreover, if $S = \varnothing$, then for every $f \in \mathcal{F}$ there exists an $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k})$ such that $\widehat{f} = f$ on $\mathbf{T}_{N,k}$ and $\widehat{f}(\widehat{\mathbf{X}}_{N,k}) \subset f(\mathbf{T}_{N,k})$. In this case the assumption that D_j's are domains of holomorphy is not necessary.