POLYNOMIAL CONVEXITY, POSITIVE CURRENTS, AND POLETSKY DISCS

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Let $K \subset \mathbb{C}^n$ be a compact set. Define its *polynomial hull*:

$$\widehat{K} := \{z \in \mathbb{C}^n : |p(z)| \leq \sup_K |p| \text{ for every polynomial } p\}.$$

Let $\Omega \subset \mathbb{C}^n$ be a domain. A probability measure μ with $\operatorname{supp}\mu \subset\subset \Omega$ is called a *Jensen measure* (for $PSH(\Omega)$) at x if

$$\varphi(x) \leq \int \varphi_\Omega \, d\mu$$

for any $PSH(\Omega)$ (and we write $\mu \in J_x(\Omega)$).

Theorem 1. *For a compact set $K \subset \mathbb{C}^n$ and a point $x \notin K$ the following conditions are equivalent:*

(a) $x \in \widehat{K}$*;*
(b) *there exists a probability measure μ with* $\operatorname{supp}\mu \subset K$ *such that for every polynomial p*
$$\log |p(x)| \leq \int \log |p| \, d\mu;$$
(c) *there exists a probability measure μ with* $\operatorname{supp}\mu \subset K$ *such that for $\varphi \in$* $\mathrm{PSH}(\mathbb{C}^n)$
$$\varphi(x) \leq \int \varphi \, d\mu;$$
(d) *there exists a nonnegative current of bidegree $(n-1, n-1)$ with compact support such that $x \in \operatorname{supp} T$ and*
$$dd^c T = \mu - \delta_x$$
for some Jensen measure $\mu \in J_x(\mathbb{C}^n)$ and the Dirac measure δ_x;
(e) *there exists a (or for any) Runge domain $\Omega \supset K$ the following holds: for every open set $U \supset K$ and every $\varepsilon > 0$ there is a holomorhic disc $f : \overline{\mathbb{D}} \to \Omega$ satisfying $f(0) = x$ and*
$$m(\{\lambda \in \partial\mathbb{D} : f(\lambda) \notin U\}) < \varepsilon,$$
where m denotes Lebesgue measure on the boundary of the unit disc $\mathbb{D} \subset \mathbb{C}$.

The equivalence (a) $\Longleftrightarrow$ (d) is due to Duval and Sibony ([DS]), while (a) $\Longleftrightarrow$ (e) was proven by Poletsky ([P]). Wold ([W]) showed that (d) follows easily from (e). (For (a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c) see [S].)

References

[DS] J. Duval, N. Sibony, *Polynomial convexity, rational convexity, and currents*, Duke Math. J. 79 (1995), 487–513.

[P] E. A. Poletsky, *Holomorphic currents*, Indiana Univ. Math. J. 42 (1993), 85–144.

[S] N. Sibony, *Prolongements des fonctions holomorphes bornée et métrique de Carathéodory*, Invent. Math. 29 (1975), 205–230.

[W] E. F. Wold, *A note on polynomial convexity: Poletsky disks, Jensen measures and positive currents*, J. Geom. Anal. (published online: 23 April 2010).

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