RIGID CHARACTERIZATIONS OF PSEUDOCONVEX DOMAINS

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Abstract. We proof that an open set D in $\mathbb{C}^n$ is pseudoconvex if and only if for any $z \in D$ the largest balanced domain centered at z and contained in D is pseudoconvex, and consider analogues of that characterization in the linearly convex case.

Introduction.

The main purpose of this note is to characterize the pseudoconvexity of an open set D in $\mathbb{C}^n$ in terms of pseudoconvexity of $B_{D,z}$, $z \in D$, i.e. in terms of pseudoconvexity in the "vertical" directions of H_D.

First let the balanced indicatrix of D at z as

$$I_{D,z} = \{X \in \mathbb{C}^n : z + \lambda X \in D, if|\lambda| \leq 1\}.$$

Then we have

$$H_D = \{(z, w) \in D \times \mathbb{C}^n : w \in I_{D,z}\}.$$

This note is based on the follow theorem

Theorem 1. *If an open set D in $\mathbb{C}^n$ is not pseudoconvex, then there is a point $a \in \partial D$, say the origin and a real-valued quadratic polynomial q such that $q(a) = 0, \partial q(a) \neq 0$,*

$$\sum_{j,k=1}^{n} \frac{\partial^2 q}{\partial z_j \partial \overline{z_k}} X_j \overline{X_k} < 0$$

for some vector $X \in \mathbb{C}^n$ with $< \partial q(a), X >= 0$, and D contains the set $\{q < 0\}$ near a.

Therefore, after an affine change of coordinates, we may assume $0 \in \partial D$ and, near this point, D contains the set

$$\{a \in \mathbb{C}^n : 0 > Rez_1 + (Imz_1)^2 + |z_2|^2 + \cdots + |z_{n-1}|^2 + c(Imz_n)^2 - (Rez_n)^2\},$$

where $-1 < c < 1$.

Then we have our main result

Theorem 2. *If D is a proper open subset of $\mathbb{C}^n$. Then$(1) \Leftrightarrow (2) \Leftrightarrow (3)$:*

(1) D is pseudoconvex;
(2) H_D is pseudoconvex;
(3) $I_{D,z}$ is pseudoconvex for any $z \in D$.

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