RIGID CHARACTERIZATIONS OF PSEUDOCONVEX DOMAINS

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Abstract. We prove that an open set $D$ in $\mathbb{C}^n$ is pseudoconvex if and only if for any $z \in D$ the largest balanced domain centered at $z$ and contained in $D$ is pseudoconvex, and consider analogues of that characterization in the linearly convex case.

Introduction.

The main purpose of this note is to characterize the pseudoconvexity of an open set $D$ in $\mathbb{C}^n$ in terms of pseudoconvexity of $B_{D,z}$, $z \in D$, i.e. in terms of pseudoconvexity in the "vertical" directions of $H_D$.

First let the balanced indicatrix of $D$ at $z$ as

$$I_{D,z} = \{ X \in \mathbb{C}^n : z + \lambda X \in D, |\lambda| \leq 1 \}.$$  

Then we have

$$H_D = \{ (z, w) \in D \times \mathbb{C}^n : w \in I_{D,z} \}.$$  

This note is based on the following theorem

**Theorem 1.** If an open set $D$ in $\mathbb{C}^n$ is not pseudoconvex, then there is a point $a \in \partial D$, say the origin and a real-valued quadratic polynomial $q$ such that $q(a) = 0$, $\partial q(a) \neq 0$,

$$\sum_{j,k=1}^{n} \frac{\partial^2 q}{\partial z_j \partial z_k} X_j X_k < 0$$

for some vector $X \in \mathbb{C}^n$ with $< \partial q(a), X > = 0$, and $D$ contains the set $\{ q < 0 \}$ near $a$.

Therefore, after an affine change of coordinates, we may assume $0 \in \partial D$ and, near this point, $D$ contains the set

$$\{ a \in \mathbb{C}^n : 0 > Re z_1 + (Im z_1)^2 + |z_2|^2 + \cdots + |z_{n-1}|^2 + c(Im z_n)^2 - (Re z_n)^2 \},$$

where $-1 < c < 1$.

Then we have our main result

**Theorem 2.** If $D$ is a proper open subset of $\mathbb{C}^n$. Then $(1) \Leftrightarrow (2) \Leftrightarrow (3)$:

1. $D$ is pseudoconvex;
2. $H_D$ is pseudoconvex;
3. $I_{D,z}$ is pseudoconvex for any $z \in D$.