The particular property of conical plurisubharmonic measure and some of it's applications

In the following definitions we assume that D is an open subset of $\mathbb{C}^n$ and that A is a subset of ∂D.

Def. 1. We say that D is locally $\mathcal{C}^2$-*smooth* on A, if for every $\zeta \in A$ there exists an open neighborhood U of ζ and a function $\rho \in \mathcal{C}^2(U)$ such that $d\rho(\zeta) \neq 0$ and

$$D \cap U = \{z \in U : \ \rho(\zeta) < 0\}.$$

Def. 2. For any $\zeta \in A$ and for every $\alpha \in (1, \infty)$ we define *a conical approach region* $\mathcal{A}_\alpha(\zeta)$

$$\mathcal{A}_\alpha(\zeta) := \{z \in D : \ |z - \zeta| < \alpha \ \mathrm{dist}(z, T_\zeta)\},$$

where T_ζ denotes a hyperplane tangent to ∂D at ζ.

Def. 3. For any $u : D \to \mathbb{R}$ we define

$$\widehat{u}(z) := \begin{cases} u(z) & \text{for} \quad z \in D \\ \sup\limits_{\alpha > 1} \limsup\limits_{w \in \mathcal{A}_\alpha(\zeta), \ w \to z} u(w) & \text{for} \quad z \in \partial D \end{cases}.$$

Def. 4. We define *a conical plurisubharmonic measure of A relatively D* as

$$\omega(z, A, D) := h^*_{A,D}(z),$$

where

$$h_{A,D} := \sup\{u \in \mathcal{PSH}(D) : \ u \le 1 \ on \ D, \ \widehat{u} \le 0 \ on \ A\}.$$

Def. 5. We say that a manifold $M \subset \mathbb{C}^n$ of class $\mathcal{C}^2$ is *generic*, if for every $z \in M$ a complex linear hull of $T_z M$ coincides with $\mathbb{C}^n$.

Def. 6. We say that a smooth, generic manifold $M \subset \mathbb{C}^n$ of class $\mathcal{C}^2$ is *totally real*, if $\dim_{\mathbb{R}} M = n$.

Def. 7. We say that $z \in \mathbb{C}^n$ is *a density point relative to A*, if

$$\mathrm{d}(z) := \lim_{r \to 0} \frac{\mathrm{m}(A \cap \mathbb{B}_r(z))}{\mathrm{m}(\mathbb{B}_r(z))} = 1.$$

Main Theorem. *Let $M \subset \mathbb{C}^n$ be a generic manifold of class $\mathcal{C}^2$, let $D \subset \mathbb{C}^n$ be a domain such that $M \subset \partial D$ and D is locally $\mathcal{C}^2$-smooth on M. Let $A \subset M$ be a measurable subset of positive measure. Then for all density points z relative to A, $\widehat{\omega}(z, A, D) = 0$.*

In the first step of the proof of Main Theorem we assume that M is a totally real manifold such that $0 \in M$, 0 is a density point relative to A and $T_0 M = \mathbb{R}^n$. We will need the following:

Theorem 1 (see [1]). *Let $M \subset \mathbb{C}^n$ be a totally real manifold of class $\mathcal{C}^2$ such that $0 \in M$ and in a neighborhood $\mathcal{V}$ of $0 \in \mathbb{C}^n$ it is given by the equation $z = x + ih(x)$, where h is a function of class $\mathcal{C}^2$ defined in $\mathcal{U}'$ - a neighborhood of $0 \in \mathbb{R}^n$ with values in $\mathbb{R}^n$ such that $h(0) = 0$ and $dh(0) = 0$.*
For a cone Γ with vertex at 0 define $W_{\Gamma,\mathcal{V}} := \{z \in \mathcal{V} : \ \mathrm{Im}\ z - h(\mathrm{Re}\ z) \in \Gamma\}$.
Let $\Gamma' \subset \Gamma''$ be two open cones in $\mathbb{R}^n$ with vertices at 0 such that $\Gamma' \cap \mathbb{S}^{n-1} \subset\subset \Gamma'' \cap \mathbb{S}^{n-1}$ and $W_{\Gamma'',\mathcal{V}} \subset D$.
Let $\varphi \in \mathcal{C}^\infty(\overline{\Delta}) \cap \mathcal{H}(\Delta)$ be such that $\varphi = 0$ on $\{e^{i\theta} \in \partial\Delta : \ |\theta| \le \frac{\pi}{2}\}$, $\varphi < 0$ on the rest of

$\partial\Delta$ and $\frac{\partial(\varphi\circ u)}{\partial s}(s,\theta)|_{s=1,\theta=0}=1$, where $u:[0,1]\times[-\pi,\pi]\ni(s,\theta)\mapsto se^{i\theta}\in\overline{\Delta}$.

Then there exists an open neighborhood $\mathcal{U}\subset\mathcal{U}'$ of $0\in\mathbb{R}^n$, an open neighborhood U of $0\in\mathbb{R}^{n-1}$ and open neighborhood V of $0\in\mathbb{R}^n$, a function $G\in\mathcal{C}^1(\overline{\Delta}\times U\times V)$ with values in $\overline{W}_{\Gamma'',\mathcal{V}}$ such that $G(\cdot,\zeta,\tau)\in\mathcal{O}(\Delta)\ \forall(\zeta,\tau)\in U\times V$, an open arc $\gamma\subset\partial\Delta$ with $1\in\gamma$ and $\beta\in\mathbb{R}\setminus\{0\}$ such that the following conditions are satisfied:

(i) for every $(\zeta,\tau)\in U\times V$ the holomorphic disc $G(\cdot,\zeta,\tau)$ is attached to M on γ, moreover, $G(\cdot,0,0)=0$ on $\overline{\Delta}$,

(ii) for every $\tau\in V$ $G_\tau:(s,\theta,\zeta)\mapsto G(se^{i\theta},\zeta,\tau)$ defines for each fixed $s\in[\frac{1}{2},1]$ a diffeomorphism of $\{se^{i\theta}:\ \theta\in\gamma\}\times U$ into $\mathbb{C}^n$, in particular it maps $\{\theta:\ e^{i\theta}\in\gamma\}\times U$ onto an open neighborhood of 0 in M. Moreover, for $e^{i\theta}\in\gamma,\ s\in(0,1),\ \zeta\in U,\ |\zeta|\leq 4|\tau|$

$$\frac{\partial \mathrm{Re}\ G_\tau(s,\theta,\zeta)}{\partial(\theta,\zeta)}=\beta\frac{\partial(\varphi\circ u)(s,\theta)}{\partial s}|\tau|\cdot\mathrm{Id}_\theta+\mathrm{Id}_\zeta+o(|\tau|),$$

(iii) there exists a conformal map ψ which maps Δ onto a Jordan domain $E\subset\Delta$ with smooth boundary such that $\gamma\subset\partial E$, $\psi(1)=1$ and for $e^{i\theta}\in\gamma,\ s\in(0,1),\ \zeta\in U,\ |\zeta|\leq 4|\tau|$

$$\frac{\partial \mathrm{Im}\ G(u(s,\theta),\zeta,\tau)}{\partial\tau}=\varphi(\psi(u(s,\theta)))\cdot\mathrm{Id}_\tau+o(\tau).$$

References

[1] B. Coupet, *Construction de disques analytiques et régularité de fonctions holomorphes au bord*, Math. Z. 209 (2), 1992, pp 179 - 204.