A HARTOGS-TYPE EXTENSION THEOREM FOR GENERALIZED 
(N,K)-CROSSES WITH PLURIPOLAR SINGULARITIES

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Abstract. The aim of the talk is to present an extension theorem for (N,k)-crosses with pluripolar singularities.

1. Definitions and main results

Definition 1.1 (Relative extremal function). Let \( X \) be a Riemann domain over \( \mathbb{C}^n \) and let \( A \subset X \). The relative extremal function of \( A \) with respect to \( X \) is a function
\[
h_{A,X} := \sup \{ u \in \mathcal{P}\mathcal{S}\mathcal{H}(X) : u \leq 1, u|_A \leq 0 \}.
\]
For an open set \( Y \subset X \) we define \( h_{A,Y} := h_{A \cap Y,Y} \).

Definition 1.2. Let \( D \) be a Riemann domain over \( \mathbb{C}^n \). A set \( A \subset D \) is called pluriregular at a point \( a \in A \) if \( h_{A,U} = 0 \) for any open neighborhood \( U \) of the point \( a \).
We call \( A \) locally pluriregular if \( A \neq \emptyset \) and \( A \) is pluriregular at every point \( a \in A \).

Let \( D_j \) be a Riemann domain over \( \mathbb{C}^n \) and let \( A_j \subset D_j \) be locally pluriregular, \( j = 1, \ldots , N \), where \( N \geq 2 \). For \( \alpha = (\alpha_1, \ldots , \alpha_N) \in \{0,1\}^N \) define:
\[
\mathcal{X}_\alpha := \mathcal{X}_{1,\alpha_1} \times \ldots \times \mathcal{X}_{N,\alpha_N},
\]
\[
\mathcal{X}_{j,\alpha_j} := \begin{cases} D_j & \text{when } \alpha_j = 1, \\ A_j & \text{when } \alpha_j = 0, \end{cases} j = 1, \ldots , N.
\]

When no confusion arise we will use the following convention: for \( A \subset \prod_{j \in I} D_j \) and \( B \subset \prod_{j \in J} D_j \), where \( I \cup J = \{1, \ldots , N\} \), by \( A \times B \) we will denote a product \( C_1 \times \ldots \times C_N \), where
\[
C_j = \begin{cases} A_j & \text{for } j \in I, \\ B_j & \text{for } j \in J. \end{cases}
\]

To simplify the notation let us define families
\[
\mathcal{I}_k^N := \{ \alpha \in \{0,1\}^N : |\alpha| = k \}, \quad \mathcal{J}_k^N := \{ \alpha \in \{0,1\}^N : 1 \leq |\alpha| \leq k \}.
\]

Definition 1.3. For a \( k \in \{1, \ldots , N\} \) we define an \( (N,k) \)-cross
\[
\mathcal{X}_{N,k} = \mathcal{X}_{N,k}((A_j, D_j)_{j=1}^N) := \bigcup_{\alpha \in \mathcal{I}_k^N} \mathcal{X}_\alpha.
\]

For \( \alpha \in \mathcal{J}_k^N \) let \( \Sigma_\alpha \subset \prod_{j: \alpha_j = 0} A_j \) and put
\[
\mathcal{X}_\alpha^{\Sigma} := \{ z \in \mathcal{X}_\alpha : z_\alpha \not\in \Sigma_\alpha \}, \quad \alpha \in \mathcal{J}_k^N,
\]
where \( z_\alpha \) denotes a projection of \( z \) on \( \prod_{j: \alpha_j = 0} D_j \).
Definition 1.4. We define a generalized \((N,k)\)-cross \(T_{N,k}\)

\[
T_{N,k} = \bigcup_{\alpha \in I^N_k} \mathcal{X}_\alpha^\Sigma
\]

and a generalized \((N,k)\)-cross \(Y_{N,k}\)

\[
Y_{N,k} = \bigcup_{\alpha \in J^N_k} \mathcal{X}_\alpha^\Sigma.
\]

For \(k = 1\) we call \(X_{N,1}((A_j, D_j)_{j=1}^N)\) an \(N\)-fold cross \(X\) and we use the following notation

\[
X = X(A_1, \ldots, A_N; D_1, \ldots, D_N) = X((A_j, D_j)_{j=1}^N) = \bigcup_{j=1}^N (A_j' \times D_j \times A_j''),
\]

where

\[
A_j' := A_1 \times \ldots \times A_{j-1}, \quad j = 2, \ldots, N,
\]

\[
A_j'' := A_{j+1} \times \ldots \times A_N, \quad j = 1, \ldots, N - 1,
\]

\[
A_1' \times D_1 \times A_1'' := D_1 \times A_1'',
\]

\[
A_N' \times D_N \times A_N'' := A_N' \times D_N.
\]

For \(\Sigma_j \subset A_j' \times A_j'', j = 1, \ldots, N\) put

\[
\mathcal{X}_j := \{(a_j', z_j, a_j'') \in A_j' \times D_j \times A_j'': (a_j', a_j'') \notin \Sigma_j\},
\]

where

\[
a_j' := (a_1, \ldots, a_{j-1}), \quad j = 2, \ldots, N,
\]

\[
a_j'' := (a_{j+1}, \ldots, a_N), \quad j = 1, \ldots, N - 1,
\]

\[
(a_1', z_1, a_1'') := (z_1, a_1'),
\]

\[
a_N', z_N, a_N'') := (a_N', z_N).
\]

We call \(T_{N,1}((A_j, D_j, \Sigma_j)_{j=1}^N) = \bigcup_{j=1}^N \mathcal{X}_j\) a generalized \(N\)-fold cross \(T\).

Definition 1.5. For \((N,k)\)-crosses we define their centers as

\[
c(X_{N,k}) := A_1 \times \ldots \times A_N,
\]

\[
c(T_{N,k}) := T_{N,k} \cap (A_1 \times \ldots \times A_N),
\]

\[
c(Y_{N,k}) := Y_{N,k} \cap (A_1 \times \ldots \times A_N).
\]

Definition 1.6. For a cross \(X_{N,k} = X_{N,k}((A_j, D_j)_{j=1}^N)\) we define its hull

\[
\hat{X}_{N,k} = \bigcup_{\alpha \in I^N_k} (A_j, D_j)_{j=1}^N := \{(z_1, \ldots, z_N) \in D_1 \times \ldots \times D_N : \sum_{j=1}^N h_{A_j, D_j}(z_j) < k\},
\]

where \(h_{B,D}\) denotes relative extremal function of \(B\) with respect to \(D\).

For an \(\alpha \in J^N_k\) and for an \(a \in (\prod_{j: \alpha_j = 0} A_j)\) define a function.
Definition 1.7. Let $T$ be holomorphic on $\prod Y$. For an $\alpha \in \mathcal{J}_{\leq N}$ and for an $a \in \prod A_j$ let $M_{a,\alpha}$ denote a fiber

$$M_{a,\alpha} := \{ z \in \prod D_j : \epsilon_a(z) \in M \}.$$  

**Definition 1.8.** Let $M \subset T_{N,k}$ be such that for all $\alpha \in \mathcal{I}_{k}^{N}$ and for all $a \in ( \prod A_j ) \setminus \Sigma_{\alpha}$ the set $( \prod D_j ) \setminus M_{a,\alpha}$ is open. A function $f : T_{N,k} \setminus M \to \mathbb{C}$ is called *separately holomorphic on* $T_{N,k} \setminus M$ ($f \in \mathcal{O}_{S}(T_{N,k} \setminus M)$), if for all $\alpha \in \mathcal{I}_{k}^{N}$ and for all $a \in ( \prod A_j ) \setminus \Sigma_{\alpha}$, the function

$$(\prod_{j : \alpha_j = 1} D_j) \setminus M_{a,\alpha} \ni z \mapsto f \circ \epsilon_a =: f_{a,\alpha}(z)$$  

is holomorphic.

For generalized $(N, k)$-cross $Y_{N,k}$ we state an analogical definition.

**Definition 1.9.** Let $M \subset Y_{N,k}$ be such that for all $\alpha \in \mathcal{J}_{\leq N}$ and for all $a \in ( \prod A_j ) \setminus \Sigma_{\alpha}$ the set $( \prod D_j ) \setminus M_{a,\alpha}$ is open. A function $f : Y_{N,k} \setminus M \to \mathbb{C}$ is called *separately holomorphic on* $Y_{N,k} \setminus M$ ($f \in \mathcal{O}_{S}(Y_{N,k} \setminus M)$), if for all $\alpha \in \mathcal{I}_{k}^{N}$ and for all $a \in ( \prod A_j ) \setminus \Sigma_{\alpha}$, the function $(*)$ is holomorphic.

For $\alpha \in \mathcal{J}_{\leq N}$ and for $b \in \prod D_j$ define a function

$$\kappa^b_{\alpha} = (\kappa^b_{\alpha,1}, \ldots, \kappa^b_{\alpha,N}) : \prod_{j : \alpha_j = 0} A_j \to \mathcal{X}_{\alpha},$$

$$\kappa^b_{\alpha,j}(z) := \begin{cases} z_j & \text{when } \alpha_j = 0, \\ b_j & \text{when } \alpha_j = 1, \end{cases} \quad j = 1, \ldots, N.$$  

Let $M \subset T_{N,k}$. For $\alpha \in \mathcal{J}_{\leq N}$ and for $b \in \prod D_j$ let $M_{b,\alpha}^a$ denote a fiber

$$M_{b,\alpha}^a := \{ z \in \prod A_j : \kappa^b_{\alpha}(z) \in M \}.$$  

**Definition 1.9.** Let $M \subset T_{N,k}$ be such that for all $\alpha \in \mathcal{I}_{k}^{N}$ and for all $a \in ( \prod A_j ) \setminus \Sigma_{\alpha}$ the set $( \prod D_j ) \setminus M_{a,\alpha}$ is open. By $\mathcal{O}_{S}(T_{N,k} \setminus M)$ we will denote a space of functions
f ∈ \mathcal{O}_S(T_{N,k} \setminus M) such that for all \alpha ∈ I^N_k and for all b ∈ (\prod_{j:a_j=1} A_j), the function

\[(\prod_{j:a_j=0} A_j) \setminus (\Sigma_\alpha \cup M^k_{b,\alpha}) \ni z \mapsto f_{b,\alpha}^k =: f_{b,\alpha}(z)\]
is continuous.

**Theorem 1.10** (Extension theorem for \((N,k)\)-crosses with pluripolar singularities). Let \(D_j\) be a Riemann domain of holomorphy over \(\mathbb{C}^{a_j}\), \(A_j ⊂ D_j\) be locally pluriregular, \(j = 1, \ldots, N\). For \(\alpha ∈ I^N_k\) let \(\Sigma_\alpha ⊂ \prod_{j:a_j=0} A_j\), be pluripolar. Let

\[
X_{N,k} := X_{N,k}((A_j, D_j)_{j=1}^N), \quad T_{N,k} := T_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha ∈ I^N_k}).
\]

Let \(M\) be a relatively closed, pluripolar subset of \(T_{N,k}\) such that for all \(\alpha ∈ I^N_k\) and all \(a ∈ (\prod_{j:a_j=0} A_j) \setminus \Sigma_\alpha\) the fiber \(M_{a,\alpha}\) is pluripolar. Let

\[
\mathcal{F} := \begin{cases} \mathcal{O}_S(X_{N,k} \setminus M), & \text{if for any } \alpha ∈ I^N_k \text{ we have } \Sigma_\alpha = \emptyset, \\ \mathcal{O}_S(T_{N,k} \setminus M), & \text{otherwise} \end{cases}
\]

Then there exists a relatively closed, pluripolar set \(\hat{M} ⊂ \hat{X}_{N,k}\) and a generalized \((N,k)\)-cross \(\hat{T}_{N,k} := T_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma'_\alpha)_{\alpha ∈ I^N_k}) \subset T_{N,k}\) with \(\Sigma_\alpha ⊂ \Sigma'_\alpha \subset \prod_{j:a_j=0} A_j\) pluripolar, \(\alpha ∈ I^N_k\), such that:

- \(\hat{M} ∩ (c(T_{N,k}) ∪ T'_{N,k}) ⊂ M\),
- for any \(f ∈ \mathcal{F}\) there exists a function \(\hat{f} ∈ \mathcal{O}(\hat{X}_{N,k} \setminus \hat{M})\) such that \(\hat{f} = f\) on \((c(T_{N,k}) ∪ T'_{N,k}) \setminus M\),
- \(\hat{M}\) is singular with respect to \(\{\hat{f} : f ∈ \mathcal{F}\}\).

**Theorem 1.11.** Let \(D_j\) be a Riemann domain of holomorphy over \(\mathbb{C}^{a_j}\), \(A_j ⊂ D_j\) be locally pluriregular, \(j = 1, \ldots, N\). For \(\alpha ∈ J^N_{≤ k}\) let \(\Sigma_\alpha\) be a pluripolar subset of \(\prod_{j:a_j=0} A_j\). Let

\[
X_{N,k} := X_{N,k}((A_j, D_j)_{j=1}^N), \quad T_{N,k} := T_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha ∈ J^N_{≤ k}}),
\]

\[
Y_{N,k} := Y_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha ∈ J^N_{≤ k}}),
\]

\[
W_{N,k} ∈ \{X_{N,k}, T_{N,k}, Y_{N,k}\}, \ M ⊂ c(W_{N,k}) \text{ and } \mathcal{F} ⊂ \{f : c(W_{N,k}) \setminus M → \mathbb{C}\} \text{ be such that:}
\]

\[(T1) \quad M \text{ is pluripolar},
\]

\[(T2) \quad \text{for any } \alpha ∈ J^N_{≤ k} \text{ and any } a ∈ (\prod_{j:a_j=0} A_j) \setminus \Sigma_\alpha \text{ the fiber } M_{a,\alpha} \text{ is pluripolar,}
\]

\[(T3) \quad \text{for any } \alpha ∈ J^N_{≤ k} \text{ and any } a ∈ (\prod_{j:a_j=0} A_j) \setminus \Sigma_\alpha \text{ there exists a closed, pluripolar set}
\]

\[
\hat{M}_{a,\alpha} ⊂ \prod_{j:a_j=1} D_j \text{ such that } \hat{M}_{a,\alpha} ∩ \prod_{j:a_j=1} A_j ⊂ M_{a,\alpha}.
\]

\[\tag{1}\]

(That is, for all \(a ∈ \hat{M}\) and \(U_a\)-open neighborhood of \(a\) there exists \(\hat{f} ∈ \{\hat{f} : f ∈ \mathcal{F}\}\) such that \(\hat{f}\) does not extend holomorphically on \(U_a\).)

\[\tag{2}\]

(Actually we can assume a bit less: \(M\) is such that for all \(j ∈ \{1, \ldots, N\}\) the set \(\{a_j ∈ A_j : M_{(\cdot, a_j, \cdot)} \text{ is not pluripolar}\}\) is pluripolar.

\[\tag{3}\]

(When \(k = N\) we assume that there exists \(\hat{M} ∈ D_1 \times \ldots \times D_N\) closed, pluripolar, such that \(\hat{M} ∩ c(W_{N,k}) ⊂ M\).)}
(T4) for any \(a \in c(W_{N,k}) \setminus M\) there exists an \(r > 0\) such that for all \(f \in F\) there exists an \(f_a \in O(P(a, r))\) with \(f_a = f\) on \(P(a, r) \cap (c(W_{N,k}) \setminus M)\).

(T5) for any \(f \in F\), any \(\alpha \in J^N_{\Sigma,k}\) and any \(a \in \bigcap_{j: \alpha_j = 0} A_j \setminus \Sigma_a\) there exists a function

\[
\tilde{f}_{a,\alpha} \in O(\prod_{j: \alpha_j = 1} D_j \setminus \tilde{M}_{a,\alpha}) \text{ such that } \tilde{f}_{a,\alpha} = f_{a,\alpha} \text{ on } \bigcap_{j: \alpha_j = 1} A_j \setminus M_{a,\alpha}.\tag{4}
\]

Then there exists a relatively closed, pluripolar set \(\tilde{M} \subset \tilde{X}_{N,k}\) such that:

- \(\tilde{M} \cap c(W_{N,k}) \subset M\),
- for any \(f \in F\) there exists \(\hat{f} \in O(\tilde{X}_{N,k} \setminus \tilde{M})\) such that \(\hat{f} = f\) on \(c(W_{N,k}) \setminus M\),
- \(\tilde{M}\) is singular with respect to \(\{\hat{f} : f \in F\}\),
- if for all \(\alpha \in J^N_{\Sigma,k}\) and all \(a \in \bigcap_{j: \alpha_j = 0} A_j \setminus \Sigma_a\) we have \(\tilde{M}_{a,\alpha} = \emptyset\), then \(\tilde{M} = \emptyset\),
- if for all \(\alpha \in J^N_{\Sigma,k}\) and all \(a \in \bigcap_{j: \alpha_j = 0} A_j \setminus \Sigma_a\) the set \(\tilde{M}_{a,\alpha}\) is thin in \(\bigcap_{j: \alpha_j = 1} D_j\), then \(\tilde{M}\) is analytic in \(\tilde{X}_{N,k}\).

**Proposition 1.12.** Let \(D_j, A_j\) and \(\Sigma_a\) be as in Theorem 1.11. Let

\[
\begin{align*}
X_{N,k} &:= X_{N,k}((A_j, D_j)_{j=1}^N), & T_{N,k} &:= T_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_a)_{\alpha \in J^N_{\Sigma,k}}), \\
Y_{N,k} &:= Y_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_a)_{\alpha \in J^N_{\Sigma,k}}),
\end{align*}
\]

\(W_{N,k} \subset \{X_{N,k}, T_{N,k}, Y_{N,k}\}\). Let \(M \subset W_{N,k}\) and \(F \subset O_S(W_{N,k} \setminus M)\) be such that:

- (P1) \(M \cap c(W_{N,k})\) is pluripolar,
- (P2) for any \(\alpha \in J^N_{\Sigma,k}\) and any \(a \in \bigcap_{j: \alpha_j = 0} A_j \setminus \Sigma_a\) the fiber \(M_{a,\alpha}\) is pluripolar and relatively closed in \(\bigcap_{j: \alpha_j = 1} D_j\),
- (P3) for any \(a \in c(W_{N,k}) \setminus M\) there exists an \(r > 0\) such that for all \(f \in F\) there exists a function \(f_a \in O(P(a, r))\) with \(f_a = f\) on \(P(a, r) \cap (c(W_{N,k}) \setminus M)\).

Then there exists a relatively closed, pluripolar set \(\tilde{M} \subset \tilde{X}_{N,k}\) such that:

- \(\tilde{M} \cap c(W_{N,k}) \subset M\),
- for any \(f \in F\) there exists \(\hat{f} \in O(\tilde{X}_{N,k} \setminus \tilde{M})\) such that \(\hat{f} = f\) on \(c(W_{N,k}) \setminus M\),
- \(\tilde{M}\) is singular with respect to \(\{\hat{f} : f \in F\}\),
- if \(M = \emptyset\), then \(\tilde{M} = \emptyset\),
- if for all \(\alpha \in J^N_{\Sigma,k}\) and all \(a \in \bigcap_{j: \alpha_j = 0} A_j \setminus \Sigma_a\) the fiber \(M_{a,\alpha}\) is thin in \(\bigcap_{j: \alpha_j = 1} D_j\), then \(\tilde{M}\) is analytic in \(\tilde{X}_{N,k}\).

2. **Prerequisites**

**Theorem 2.1** (see [JarPfl 2007], Theorem 1.1). Let \(D_j\) be a Riemann domain of holomorphy over \(\mathbb{C}^n\), \(A_j \subset D_j\) be locally pluriregular and let \(\Sigma_j \subset A'_j \times A''_j\) be pluripolar, \(j = 1, \ldots, N\). Put

\[
X := X((A_j, D_j)_{j=1}^N), & T := T((A_j, D_j, \Sigma_j)_{j=1}^N).
\]

Let \(F \subset \{f : c(T) \setminus M \rightarrow \mathbb{C}\}\) and let \(M \subset T\) be such that:

- for any \(j \in \{1, \ldots, N\}\) an any \((a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j\) the fiber \(M(a'_j, a''_j)\) is pluripolar.

\[(4)\text{When } k = N \text{ we assume that there exists } \tilde{f} \in O(D_1 \times \ldots \times D_N \setminus \tilde{M}) \text{ such that } \tilde{f} = f \text{ on } c(W_{N,k}) \setminus M.\]
• for any \( j \in \{1, \ldots, N\} \) and any \( (a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j \) there exists a closed, pluripolar set \( \tilde{M}_{a,j} \subset D_j \) such that \( \tilde{M}_{a,j} \cap A_j \subset M(a'_j, a''_j) \),
• for any \( a \in c(T) \setminus M \) there exists an \( r > 0 \) such that for all \( f \in \mathcal{F} \) there exists an \( f_a \in \mathcal{O}(P(a, r)) \) with \( f_a = f \) on \( P(a, r) \cap (c(T) \setminus M) \),
• for any \( f \in \mathcal{F} \), any \( j \in \{1, \ldots, N\} \), and any \( (a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j \) there exists a function \( \tilde{f}_{a,j} \in \mathcal{O}(D_j \setminus \tilde{M}_{a,j}) \) such that \( \tilde{f}_{a,j} = f(a'_j, \cdot, a''_j) \) on \( A_j \setminus M_{a,j} \).

Then there exists a relatively closed, pluripolar set \( \tilde{M} \subset X \) such that:
• \( \tilde{M} \cap c(T) \subset M \),
• for any \( f \in \mathcal{F} \) there exists a function \( \hat{f} \in \mathcal{O}(X \setminus \tilde{M}) \) such that \( \hat{f} = f \) on \( c(T) \setminus M \),
• \( \tilde{M} \) is singular with respect to \( \{ \hat{f} : f \in \mathcal{F} \} \),
• if for all \( j \in \{1, \ldots, N\} \) and all \( (a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j \) we have \( \tilde{M}_{a,j} = \emptyset \), then \( \tilde{M} = \emptyset \),
• if for all \( j \in \{1, \ldots, N\} \) and all \( (a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j \) the set \( \tilde{M}_{a,j} \) is thin in \( D_j \), then \( \tilde{M} \) is analytic in \( X \).

Lemma 2.2 (JarPfl 2010, Lemma 4). Let \( D_j \) be a Riemann domain of holomorphy over \( \mathbb{C}^{n_j} \) and \( A_j \subset D_j \) be locally pluriregular, \( j = 1, \ldots, N \). Then for all \( z = (z_1, \ldots, z_N) \in \hat{X}_{N,k} \) we have:

\[
h_{\hat{X}_{N,k-1}, \hat{X}_{N,k}}(z) = \max \{0, \sum_{j=1}^{N} h_{A_j, D_j}(z_j) - k + 1\}.
\]

Theorem 2.3 (Cross theorem for \((N, k)\)-crosses, cf. JarPfl 2011, Theorem 7.2.7). Let \( D_j \) be a Riemann domain of holomorphy over \( \mathbb{C}^{n_j} \) and \( A_j \subset D_j \) be locally pluriregular, \( j = 1, \ldots, N \). For \( k \in \{1, \ldots, N\} \) let \( X_{N,k} := X_{N,k}((A_j, D_j)_{j=1}^{N}) \). Then for every \( f \in \mathcal{O}_S(X_{N,k}) \) there exists a unique function \( \hat{f} \in \mathcal{O}(\hat{X}_{N,k}) \) such that \( \hat{f} = f \) on \( X_{N,k} \) and \( \hat{f}(\hat{X}_{N,k}) \subset f(X_{N,k}) \).

Theorem 2.4 (Cross theorem for generalized \((N, k)\)-crosses). Let \( D_j \) be a Riemann domain over \( \mathbb{C}^{n_j} \), \( A_j \subset D_j \) be pluriregular, \( j = 1, \ldots, N \). For \( \alpha \in \mathcal{T}_k \) let \( \Sigma_\alpha \) be a subset of \( \prod_{j: \alpha_j = 0} A_j \). Let

\[
X_{N,k} := X_{N,k}((A_j, D_j)_{j=1}^{N}), \quad T_{N,k} := T_{N,k}((A_j, D_j)_{j=1}^{N}, (\Sigma_\alpha)_{\alpha \in \mathcal{T}_k}).
\]

Then for every \( f \in \mathcal{O}_S(T_{N,k}) \) there exists \( \hat{f} \in \mathcal{O}(\hat{X}_{N,k}) \) such that \( \hat{f} = f \) on \( T_{N,k} \) and \( \hat{f}(\hat{X}_{N,k}) \subset f(T_{N,k}) \).

3. Sketch of proof of Theorem 1.11

Lemma 3.1. Theorem 1.11 with \( W_{N,k} = X_{N,k} \) implies Theorem 1.11 with \( W_{N,k} \in \{T_{N,k}, Y_{N,k}\} \).

Sketch of proof of Theorem 1.11 with \( W_{N,k} = X_{N,k} \).

Step 1. Theorem 1.11 is true for any \( N \) when \( k = 1 \) (Theorem 2.1) and when \( k = N \) (in this case we assumed the thesis).

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(5) \( P(a, r) \) denotes a polydisc in Riemann domain \( D_1 \times \ldots \times D_N \) centered at \( a \) with radius \( r \).
Step 2. In particular, theorem is true for \( N = 2, k = 1, 2 \). Assume we already have Theorem 1.11 for \( (N-1, k) \), where \( k \in \{1, \ldots, N-1\} \) and for \( (N, 1), \ldots, (N, k-1) \), where \( k \in \{2, \ldots, N-1\} \). We need to prove it for \( (N, k) \).

Step 3. Fix \( s \in \{1, \ldots, N\} \) (to simplify the notation let \( s = N \)). Let
\[
Q_N := \{a_N \in A_N : M_{(-a_N)} \text{ is not pluripolar}\}.
\]
Then \( Q_N \) is pluripolar. Define
\[
X_{N-1,k}^{(s)} := X_{N-1,k}((A_j, D_j)_{j=1,j\neq s}^N), \quad s = 1, \ldots, N,
\]
in particular
\[
X_{N-1,k}^{(N)} := X_{N-1,k} := X_{N-1,k}((A_j, D_j)_{j=1}^{N-1}).
\]
Fix \( a_N \in A_N \setminus Q_N \) and define a family \( \{f(\cdot, a_N) : f \in \mathcal{F}\} \subset \{f : c(X_{N-1,k}) \to \mathbb{C}\} \).
Then from inductive assumption we get a relatively closed pluripolar set \( \widetilde{M}_{a_N} \subset X_{N-1,k} \) such that:

- \( \widetilde{M}_{a_N} \cap c(X_{N-1,k}) \subset M_{(-a_N)} \),
- for any \( f \in \mathcal{F} \) there exists \( \hat{f}_{a_N} \in \mathcal{O}(\mathcal{X}_{N-1,k} \setminus \widetilde{M}_{a_N}) \) such that \( \hat{f}_{a_N} = f(\cdot, a_N) \) on \( c(X_{N-1,k}) \setminus M_{(-a_N)} \),
- \( \widetilde{M}_{a_N} \) is singular with respect to \( \{\hat{f}_{a_N} : f \in \mathcal{F}\} \),
- if for all \( \alpha' \in \{0,1\}^{N-1} : |\alpha'| \leq k \) and all \( a' \in (\prod_{j:|\alpha_j|=0} A_j) \setminus \Sigma_{\alpha'} \), we have \( \widetilde{M}_{a',\alpha'} = \emptyset \),
- then \( \widetilde{M}_{a_N} = \emptyset \),
- if for all \( \alpha' \in \{0,1\}^{N-1} : |\alpha'| \leq k \) and all \( a' \in (\prod_{j:|\alpha_j|=0} A_j) \setminus \Sigma_{\alpha'} \), the set \( \widetilde{M}_{a',\alpha'} \) is thin in \( \prod_{j:|\alpha_j|=0} D_j \), then \( \widetilde{M}_{a_N} \) is analytic in \( X_{N-1,k} \).

Define a new cross
\[
Z_N := X(c(X_{N-1,k}), A_N; \mathcal{X}_{N-1,k}, D_N).
\]
Observe that \( Z_N \) with original \( M, (\Sigma_{\alpha})_{\alpha} \) and the family \( \mathcal{F} \) satisfies all the assumptions of Theorem 1.11 with \( N = 2, k = 1 \). Then there exists an \( \widehat{M}_N \subset Z_N \), relatively closed, pluripolar, such that:

- \( \widehat{M}_N \cap c(X_{N,k}) \subset M \),
- for any \( f \in \mathcal{F} \) there exists \( \hat{f}_N \in \mathcal{O}(Z_N \setminus \widehat{M}_N) \) such that \( \hat{f}_N = f \) on \( c(X_{N,k}) \setminus M \),
- \( \widehat{M}_N \) is singular with respect to \( \{\hat{f}_N : f \in \mathcal{F}\} \),
- if for all \( a' \in c(X_{N-1,k}) \setminus \Sigma_{(0,\ldots,0,1)} \) we have \( \widehat{M}_{a'} = \emptyset \) and for all \( a_N \in A_N \setminus Q_N \) we have \( \widehat{M}_{a_N} = \emptyset \), then \( \widehat{M}_N = \emptyset \),
- if for all \( a' \in c(X_{N-1,k}) \setminus \Sigma_{(0,\ldots,0,1)} \) the set \( \widehat{M}_{a'} \) is thin in \( D_N \) and for all \( a_N \in A_N \setminus Q_N \) the set \( \widehat{M}_{a_N} \) is thin in \( \mathcal{X}_{N-1,k} \), then \( \widehat{M}_N \) is analytic in \( Z_N \).

We repeat the reasoning above for all \( s = 1, \ldots, N-1 \), obtaining a family of functions \( \{\hat{f}_s\}_{s=1}^N \) such that for any \( s \in \{1, \ldots, N\} \) we have \( \hat{f}_s = f \) on \( c(X_{N,k}) \setminus M \). Define a new function
\[
F_f(z) := \begin{cases} 
\hat{f}_1(z) & \text{for } z \in \mathcal{Z}_1 \setminus \widehat{M}_1 \\
\vdots \\
\hat{f}_N(z) & \text{for } z \in \mathcal{Z}_N \setminus \widehat{M}_N 
\end{cases}
\]
Lemma 3.2. Function $F_f$ is well defined and holomorphic on $\left( \bigcup_{s=1}^{N} Z_s \right) \setminus \left( \bigcup_{s=1}^{N} \tilde{M}_s \right)$.

Step 4. Define a 2-fold cross

$$Z := X(X_{N-1,k-1}, A_N; \tilde{X}_{N-1,k}, D_N) \subset \bigcup_{s=1}^{N} Z_s,$$

a pluripolar set

$$\tilde{M} := \left( \bigcup_{s=1}^{N} \tilde{M}_s \right) \cap (X_{N-1,k-1} \times A_N)$$

and a family

$$\tilde{F} := \{ \tilde{f} := F_f|_{(X_{N-1,k-1} \times A_N) \setminus \tilde{M}} : f \in F \}.$$

Then $Z$, $\tilde{M}$ and $\tilde{F}$ satisfy the assumptions of Theorem 1.11 with $N = 1$ and $k = 1$. Now from Theorem 1.11 there exists a relatively closed, pluripolar set $\tilde{M} \subset \tilde{Z} = \tilde{X}_{N,k}$ such that:

- $\tilde{M} \cap (X_{N-1,k-1} \times A_N) \subset \tilde{M}$, in particular, $\tilde{M} \cap c(X_{N,k}) \subset \tilde{M}$,
- for any $\tilde{f} \in \tilde{F}$ there exists $\tilde{f} \in O(\tilde{X}_{N,k} \setminus \tilde{M})$ such that $\tilde{f} = \tilde{f}$ on $(X_{N-1,k-1} \times A_N) \setminus \tilde{M}$, in particular $\tilde{f} = f$ on $c(X_{N,k}) \setminus \tilde{M}$,
- $\tilde{M}$ is singular with respect to $\{ \tilde{f} : f \in F \}$,
- if for all $z' \in X_{N-1,k-1} \setminus P \tilde{M}' = \emptyset$ and for all $a_N \in A_N \setminus Q \tilde{M}_{a_N} = \emptyset$, then $\tilde{M} = \emptyset$,
- if for all $z' \in X_{N-1,k-1} \setminus P$ the set $\tilde{M}'$ is thin in $D_N$ and for all $a_N \in A_N \setminus Q$ the set $\tilde{M}_{a_N}$ is thin in $\tilde{X}_{N-1,k}$, then $\tilde{M}$ is analytic in $\tilde{Z}$.

Sketch of proof of Lemma 3.2. Fix $s$ and $p$. We want to show that $\tilde{f}_s = \tilde{f}_p$ on $(Z_s \cap Z_p) \setminus (\tilde{M}_s \cup \tilde{M}_p)$. To simplify the notation we may assume that $s = N - 1$ and $p = N$.

Step 1. Every connected component of $Z_{N-1} \cap Z_N$ contains part of the center.
Step 2. One connected component of $Z_{N-1} \cap Z_N$ contains whole $Z_{N-1} \cap Z_N$.
Step 3. Every connected component of $Z_{N-1} \cap Z_N$ with $\tilde{M}_{N-1} \cup \tilde{M}_N$ deleted is a domain, thus it is a connected component of $(Z_{N-1} \cap Z_N) \setminus (\tilde{M}_{N-1} \cup \tilde{M}_N)$.
Step 4. One connected component of $(Z_{N-1} \cap Z_N) \setminus (\tilde{M}_{N-1} \cup \tilde{M}_N)$ contains whole set $(Z_{N-1} \cap Z_N) \setminus (\tilde{M}_{N-1} \cup \tilde{M}_N)$.
Step 5. $\tilde{f}_{N-1} = \tilde{f}_N$ on $(Z_{N-1} \cap Z_N) \setminus (\tilde{M}_{N-1} \cup \tilde{M}_N)$.

References


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