A HARTOGS-TYPE EXTENSION THEOREM FOR GENERALIZED (N,K)-CROSSES WITH PLURIPOLAR SINGULARITIES

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ABSTRACT. The aim of the talk is to present an extension theorem for (N,k)-crosses with pluripolar singularities.

1. DEFINITIONS AND MAIN RESULTS

Definition 1.1 (Relative extremal function)**.** Let X be a Riemann domain over $\mathbb{C}^n$ and let $A \subset X$. The *relative extremal function of A with respect to X* is a function

$$\mathbf{h}_{A,X} := \sup\{u \in \mathcal{PSH}(X) : \ u \leq 1, \ u|_A \leq 0\}.$$

For an open set $Y \subset X$ we define $\mathbf{h}_{A,Y} := \mathbf{h}_{A\cap Y,Y}$.

Definition 1.2. Let D be a Riemann domain over $\mathbb{C}^n$. A set $A \subset D$ is called *pluriregular at a point $a \in \overline{A}$* if $\mathbf{h}_{A,U} = 0$ for any open neighborhood U of the point a.
We call A *locally pluriregular* if $A \neq \varnothing$ and A is pluriregular at every point $a \in A$.

Let D_j be a Riemann domain over $\mathbb{C}^{n_j}$ and let $A_j \subset D_j$ be locally pluriregular, $j = 1,\ldots,N$, where $N \geq 2$. For $\alpha = (\alpha_1,\ldots,\alpha_N) \in \{0,1\}^N$ define:

$$\mathcal{X}_\alpha := \mathcal{X}_{1,\alpha_1} \times \ldots \times \mathcal{X}_{N,\alpha_N},$$
$$\mathcal{X}_{j,\alpha_j} := \begin{cases} D_j & \text{when} \quad \alpha_j = 1 \\ A_j & \text{when} \quad \alpha_j = 0 \end{cases}, \quad j = 1,\ldots,N.$$

When no confusion arise we will use the following convetnion: for $A \subset \prod_{j\in I} D_j$ and $B \subset \prod_{j\in J} D_j$, where $I \cup J = \{1,\ldots,N\}$, by $A \times B$ we will denote a product $C_1 \times \ldots \times C_N$, where

$$C_j = \begin{cases} A_j & \text{for} \quad j \in I \\ B_j & \text{for} \quad j \in J \end{cases}.$$

To simplify the notation let us define families

$$\mathcal{I}^N_k := \{\alpha \in \{0,1\}^N : \ |\alpha| = k\}, \quad \mathcal{J}^N_{\leq k} := \{\alpha \in \{0,1\}^N : \ 1 \leq |\alpha| \leq k\}.$$

Definition 1.3. For a $k \in \{1,\ldots,N\}$ we define an (N,k)-*cross*

$$\mathbf{X}_{N,k} = \mathbb{X}_{N,k}((A_j,D_j)^N_{j=1}) := \bigcup_{\alpha\in\mathcal{I}^N_k} \mathcal{X}_\alpha.$$

For $\alpha \in \mathcal{J}^N_{\leq k}$ let $\Sigma_\alpha \subset \prod_{j:\alpha_j=0} A_j$ and put

$$\mathcal{X}^\Sigma_\alpha := \{z \in \mathcal{X}_\alpha : \ z_\alpha \notin \Sigma_\alpha\}, \quad \alpha \in \mathcal{J}^N_{\leq k},$$

where z_α denotes a projection of z on $\prod_{j:\alpha_j=0} D_j$.

Definition 1.4. We define a *generalized (N,k)-cross* $\mathbf{T}_{N,k}$

$$\mathbf{T}_{N,k} = \mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha\in\mathcal{I}_k^N}) := \bigcup_{\alpha\in\mathcal{I}_k^N} \mathcal{X}_\alpha^\Sigma$$

and a *generalized (N,k)-cross* $\mathbf{Y}_{N,k}$

$$\mathbf{Y}_{N,k} = \mathbb{Y}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha\in\mathcal{J}_{\le k}^N}) := \bigcup_{\alpha\in\mathcal{J}_{\le k}^N} \mathcal{X}_\alpha^\Sigma.$$

For $k=1$ we call $\mathbb{X}_{N,1}((A_j, D_j)_{j=1}^N)$ an *N-fold cross* $\mathbf{X}$ and we use the following notation

$$\mathbf{X} = \mathbb{X}(A_1,\dots,A_N; D_1,\dots,D_N) = \mathbb{X}((A_j, D_j)_{j=1}^N) = \bigcup_{j=1}^N (A_j' \times D_j \times A_j''),$$

where

$$\begin{aligned} &A_j' := A_1 \times \ldots \times A_{j-1},\ j = 2,\dots,N,\\ &A_j'' := A_{j+1} \times \ldots \times A_N,\ j = 1,\dots,N-1,\\ &A_1' \times D_1 \times A_1'' := D_1 \times A_1'',\\ &A_N' \times D_N \times A_N'' := A_N' \times D_N. \end{aligned}$$

For $\Sigma_j \subset A_j' \times A_j''$, $j = 1,\dots,N$ put

$$\mathcal{X}_j := \{(a_j', z_j, a_j'') \in A_j' \times D_j \times A_j'' :\ (a_j', a_j'') \notin \Sigma_j\},$$

where

$$\begin{aligned} &a_j' := (a_1,\dots,a_{j-1}),\ j = 2,\dots,N,\\ &a_j'' := (a_{j+1},\dots,a_N),\ j = 1,\dots,N-1,\\ &(a_1', z_1, a_1'') := (z_1, a_1''),\\ &(a_N', z_N, a_N'') := (a_N', z_N). \end{aligned}$$

We call $\mathbb{T}_{N,1}((A_j, D_j, \Sigma_j)_{j=1}^N) = \bigcup_{j=1}^N \mathcal{X}_j$ a *generalized N-fold cross* $\mathbf{T}$.

Definition 1.5. For (N,k)-crosses we define their *centers* as

$$\begin{aligned} &c(\mathbf{X}_{N,k}) := A_1 \times \ldots \times A_N,\\ &c(\mathbf{T}_{N,k}) := \mathbf{T}_{N,k} \cap (A_1 \times \ldots \times A_N),\\ &c(\mathbf{Y}_{N,k}) := \mathbf{Y}_{N,k} \cap (A_1 \times \ldots \times A_N). \end{aligned}$$

Definition 1.6. For a cross $\mathbf{X}_{N,k} = \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N)$ we define its *hull*

$$\widehat{\mathbf{X}}_{N,k} = \widehat{\mathbb{X}}_{N,k}((A_j, D_j)_{j=1}^N) := \Big\{(z_1,\dots,z_N) \in D_1 \times \ldots \times D_N :\ \sum_{j=1}^N \mathbf{h}_{A_j,D_j}(z_j) < k\Big\},$$

where $\mathbf{h}_{B,D}$ denotes relative extremal function of B with respect to D.

For an $\alpha \in \mathcal{J}_{\le k}^N$ and for an $a \in (\prod_{j:\alpha_j=0} A_j)$ define a function

$$\iota_\alpha^a = (\iota_{\alpha,1}^a, \dots, \iota_{\alpha,N}^a) : \prod_{j:\alpha_j=1} D_j \to \mathcal{X}_\alpha,$$

$$\iota_{\alpha,j}^a(z) := \begin{cases} z_j & \text{when} \quad \alpha_j = 1 \\ a_j & \text{when} \quad \alpha_j = 0 \end{cases}, \quad j = 1, \dots, N.$$

Let $\mathbf{W}_{N,k} \in \{\mathbf{T}_{N,k}, \mathbf{Y}_{N,k}\}$ and let $M \subset \mathbf{W}_{N,k}$. For an $\alpha \in \mathcal{J}_{\leq k}^N$ and for an $a \in \prod_{j:\alpha_j=0} A_j$ let $M_{a,\alpha}$ denote a fiber

$$M_{a,\alpha} := \{z \in \prod_{j:\alpha_j=1} D_j : \ \iota_\alpha^a(z) \in M\}.$$

Definition 1.7. Let $M \subset \mathbf{T}_{N,k}$ be such that for all $\alpha \in \mathcal{I}_k^N$ and for all $a \in (\prod_{j:\alpha_j=0} A_j) \setminus \Sigma_\alpha$ the set $(\prod_{j:\alpha_j=1} D_j\big) \setminus M_{a,\alpha}$ is open. A function $f : \mathbf{T}_{N,k} \setminus M \to \mathbb{C}$ is called *separately holomorphic on* $\mathbf{T}_{N,k} \setminus M$ ($f \in \mathcal{O}_{\mathcal{S}}(\mathbf{T}_{N,k} \setminus M)$), if for all $\alpha \in \mathcal{I}_k^N$ and for all $a \in (\prod_{j:\alpha_j=0} A_j) \setminus \Sigma_\alpha$, the function

$$(*) \qquad \big(\prod_{j:\alpha_j=1} D_j\big) \setminus M_{a,\alpha} \ni z \mapsto f \circ \iota_\alpha^a =: f_{a,\alpha}(z)$$

is holomorphic.

For generalized (N,k)-cross $\mathbf{Y}_{N,k}$ we state an analogical definition.

Definition 1.8. Let $M \subset \mathbf{Y}_{N,k}$ be such that for all $\alpha \in \mathcal{J}_{\leq k}^N$ and for all $a \in (\prod_{j:\alpha_j=0} A_j) \setminus \Sigma_\alpha$ the set $(\prod_{j:\alpha_j=1} D_j\big) \setminus M_{a,\alpha}$ is open. A function $f : \mathbf{Y}_{N,k} \setminus M \to \mathbb{C}$ is called *separately holomorphic on* $\mathbf{Y}_{N,k} \setminus M$ ($f \in \mathcal{O}_{\mathcal{S}}(\mathbf{Y}_{N,k} \setminus M)$), if for all $\alpha \in \mathcal{I}_k^N$ and for all $a \in (\prod_{j:\alpha_j=0} A_j) \setminus \Sigma_\alpha$, the function $(*)$ is holomorphic.

For $\alpha \in \mathcal{J}_{\leq k}^N$ and for $b \in \prod_{j:\alpha_j=1} D_j$ define a function

$$\kappa_\alpha^b = (\kappa_{\alpha,1}^b, \dots, \kappa_{\alpha,N}^b) : \prod_{j:\alpha_j=0} A_j \to \mathcal{X}_\alpha,$$

$$\kappa_{\alpha,j}^b(z) := \begin{cases} z_j & \text{when} \quad \alpha_j = 0 \\ b_j & \text{when} \quad \alpha_j = 1 \end{cases}, \quad j = 1, \dots, N.$$

Let $M \subset \mathbf{T}_{N,k}$. For $\alpha \in \mathcal{J}_{\leq k}^N$ and for $b \in \prod_{j:\alpha_j=1} D_j$ let $M_{b,\alpha}^\kappa$ denote a fiber

$$M_{b,\alpha}^\kappa := \{z \in \prod_{j:\alpha_j=0} A_j : \ \kappa_\alpha^b(z) \in M\}.$$

Definition 1.9. Let $M \subset \mathbf{T}_{N,k}$ be such that for all $\alpha \in \mathcal{I}_k^N$ and for all $a \in (\prod_{j:\alpha_j=0} A_j) \setminus \Sigma_\alpha$ the set $(\prod_{j:\alpha_j=1} D_j) \setminus M_{a,\alpha}$ is open. By $\mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k} \setminus M)$ we will denote a space of functions

$f \in \mathcal{O}_{\mathcal{S}}(\mathbf{T}_{N,k} \setminus M)$ such that for all $\alpha \in \mathcal{I}_k^N$ and for all $b \in (\prod_{j:\alpha_j=1} D_j)$, the function

$$(\prod_{j:\alpha_j=0} A_j) \setminus (\Sigma_\alpha \cup M_{b,\alpha}^\kappa) \ni z \mapsto f \circ \kappa_\alpha^b =: f_{b,\alpha}^\kappa(z)$$

is continuous.

Theorem 1.10 (Extension theorem for (N,k)-crosses with pluripolar singularities)**.** *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ be locally pluriregular, $j = 1, \dots, N$. For $\alpha \in \mathcal{I}_k^N$ let $\Sigma_\alpha \subset \prod_{j:\alpha_j=0} A_j$, be pluripolar. Let*

$$\mathbf{X}_{N,k} := \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N), \quad \mathbf{T}_{N,k} := \mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha \in \mathcal{I}_k^N}).$$

Let M be a relatively closed, pluripolar subset of $\mathbf{T}_{N,k}$ such that for all $\alpha \in \mathcal{I}_k^N$ and all $a \in (\prod_{j:\alpha_j=0} A_j) \setminus \Sigma_\alpha$ the fiber $M_{a,\alpha}$ is pluripolar. Let

$$\mathcal{F} := \begin{cases} \mathcal{O}_{\mathcal{S}}(\mathbf{X}_{N,k} \setminus M), & \text{if for any } \alpha \in \mathcal{I}_k^N \text{ we have } \Sigma_\alpha = \varnothing \\ \mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k} \setminus M), & \text{otherwise} \end{cases}.$$

Then there exists a relatively closed, pluripolar set $\widehat{M} \subset \widehat{\mathbf{X}}_{N,k}$ and a generalized (N,k)-cross $\mathbf{T}'_{N,k} := \mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma'_\alpha)_{\alpha \in \mathcal{I}_k^N}) \subset \mathbf{T}_{N,k}$ with $\Sigma_\alpha \subset \Sigma'_\alpha \subset \prod_{j:\alpha_j=0} A_j$, Σ'_α pluripolar, $\alpha \in \mathcal{I}_k^N$, such that:

- *$\widehat{M} \cap (c(\mathbf{T}_{N,k}) \cup \mathbf{T}'_{N,k}) \subset M$,*
- *for any $f \in \mathcal{F}$ there exists a function $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{M})$ such that $\widehat{f} = f$ on $(c(\mathbf{T}_{N,k}) \cup \mathbf{T}'_{N,k}) \setminus M$,*
- *$\widehat{M}$ is singular with respect to $\{\widehat{f} : f \in \mathcal{F}\}$[(1)].*

Theorem 1.11. *Let D_j be a Riemann domain of holomorphy over $\mathbb{C}^{n_j}$, $A_j \subset D_j$ be locally pluriregular, $j = 1, \dots, N$. For $\alpha \in \mathcal{J}_{\leq k}^N$ let Σ_α be a pluripolar subset of $\prod_{j:\alpha_j=0} A_j$. Let*

$$\mathbf{X}_{N,k} := \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N), \quad \mathbf{T}_{N,k} := \mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha \in \mathcal{I}_k^N}),$$
$$\mathbf{Y}_{N,k} := \mathbb{Y}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha \in \mathcal{J}_{\leq k}^N}),$$

$\mathbf{W}_{N,k} \in \{\mathbf{X}_{N,k}, \mathbf{T}_{N,k}, \mathbf{Y}_{N,k}\}$, $M \subset c(\mathbf{W}_{N,k})$ and $\mathcal{F} \subset \{f : c(\mathbf{W}_{N,k}) \setminus M \to \mathbb{C}\}$ be such that:

(T1) *M is pluripolar,*[(2)]

(T2) *for any $\alpha \in \mathcal{J}_{\leq k}^N$ and any $a \in (\prod_{j:\alpha_j=0} A_j) \setminus \Sigma_\alpha$ the fiber $M_{a,\alpha}$ is pluripolar,*

(T3) *for any $\alpha \in \mathcal{J}_{\leq k}^N$ and any $a \in (\prod_{j:\alpha_j=0} A_j) \setminus \Sigma_\alpha$ there exists a closed, pluripolar set $\widetilde{M}_{a,\alpha} \subset \prod_{j:\alpha_j=1} D_j$ such that $\widetilde{M}_{a,\alpha} \cap \prod_{j:\alpha_j=1} A_j \subset M_{a,\alpha}$,*[(3)]

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(1) That is, for all $a \in \widehat{M}$ and U_a-open neighborhood of a there exists $\widehat{f} \in \{\widehat{f} : f \in \mathcal{F}\}$ such that $\widehat{f}$ does not extend holomorphically on U_a.

(2) Actually we can assume a bit less: M is such that for all $j \in \{1, \dots, N\}$ the set $\{a_j \in A_j : M_{(\cdot, a_j, \cdot)} \text{ is not pluripolar}\}$ is pluripolar.

(3) When $k = N$ we assume that there exists $\widetilde{M} \in D_1 \times \ldots \times D_N$ closed, pluripolar, such that $\widetilde{M} \cap c(\mathbf{W}_{N,k}) \subset M$.

(T4) *for any* $a \in c(\mathbf{W}_{N,k}) \setminus M$ *there exists an* $r > 0$ *such that for all* $f \in \mathcal{F}$ *there exists an* $f_a \in \mathcal{O}(\mathbb{P}(a,r))$ *with* $f_a = f$ *on* $\mathbb{P}(a,r) \cap (c(\mathbf{W}_{N,k}) \setminus M)$,

(T5) *for any* $f \in \mathcal{F}$, *any* $\alpha \in \mathcal{J}^N_{\leq k}$ *and any* $a \in (\prod_{j:\alpha_j=0} A_j) \setminus \Sigma_\alpha$ *there exists a function* $\widetilde{f}_{a,\alpha} \in \mathcal{O}(\prod_{j:\alpha_j=1} D_j \setminus \widetilde{M}_{a,\alpha})$ *such that* $\widetilde{f}_{a,\alpha} = f_{a,\alpha}$ *on* $(\prod_{j:\alpha_j=1} A_j) \setminus M_{a,\alpha}$[(4)].

Then there exists a relatively closed, pluripolar set $\widehat{M} \subset \widehat{\mathbf{X}}_{N,k}$ *such that:*

- $\widehat{M} \cap c(\mathbf{W}_{N,k}) \subset M$,
- *for any* $f \in \mathcal{F}$ *there exists* $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{M})$ *such that* $\widehat{f} = f$ *on* $c(\mathbf{W}_{N,k}) \setminus M$,
- $\widehat{M}$ *is singular with respect to* $\{\widehat{f} : f \in \mathcal{F}\}$,
- *if for all* $\alpha \in \mathcal{J}^N_{\leq k}$ *and all* $a \in (\prod_{j:\alpha_j=0} A_j) \setminus \Sigma_\alpha$ *we have* $\widetilde{M}_{a,\alpha} = \varnothing$, *then* $\widehat{M} = \varnothing$,
- *if for all* $\alpha \in \mathcal{J}^N_{\leq k}$ *and all* $a \in (\prod_{j:\alpha_j=0} A_j) \setminus \Sigma_\alpha$ *the set* $\widetilde{M}_{a,\alpha}$ *is thin in* $\prod_{j:\alpha_j=1} D_j$, *then* $\widehat{M}$ *is analytic in* $\widehat{\mathbf{X}}_{N,k}$.

Proposition 1.12. *Let* D_j, A_j *and* Σ_α *be as in Theorem 1.11. Let*

$$\mathbf{X}_{N,k} := \mathbb{X}_{N,k}((A_j, D_j)_{j=1}^N), \quad \mathbf{T}_{N,k} := \mathbb{T}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha \in \mathcal{I}^N_k}),$$

$$\mathbf{Y}_{N,k} := \mathbb{Y}_{N,k}((A_j, D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha \in \mathcal{J}^N_{\leq k}}),$$

$\mathbf{W}_{N,k} \in \{\mathbf{X}_{N,k}, \mathbf{T}_{N,k}, \mathbf{Y}_{N,k}\}$. *Let* $M \subset \mathbf{W}_{N,k}$ *and* $\mathcal{F} \subset \mathcal{O}_{\mathcal{S}}(\mathbf{W}_{N,k} \setminus M)$ *be such that:*

(P1) $M \cap c(\mathbf{W}_{N,k})$ *is pluripolar,*

(P2) *for any* $\alpha \in \mathcal{J}^N_{\leq k}$ *and any* $a \in (\prod_{j:\alpha_j=0} A_j) \setminus \Sigma_\alpha$ *the fiber* $M_{a,\alpha}$ *is pluripolar and relatively closed in* $\prod_{j:\alpha_j=1} D_j$,

(P3) *for any* $a \in c(\mathbf{W}_{N,k}) \setminus M$ *there exists an* $r > 0$ *such that for all* $f \in \mathcal{F}$ *there exists a function* $f_a \in \mathcal{O}(\mathbb{P}(a,r))$ *with* $f_a = f$ *on* $\mathbb{P}(a,r) \cap (c(\mathbf{W}_{N,k}) \setminus M)$.

Then there exists a relatively closed, pluripolar set $\widehat{M} \subset \widehat{\mathbf{X}}_{N,k}$ *such that:*

- $\widehat{M} \cap c(\mathbf{W}_{N,k}) \subset M$,
- *for any* $f \in \mathcal{F}$ *there exists* $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k} \setminus \widehat{M})$ *such that* $\widehat{f} = f$ *on* $c(\mathbf{W}_{N,k}) \setminus M$,
- $\widehat{M}$ *is singular with respect to* $\{\widehat{f} : f \in \mathcal{F}\}$,
- *if* $M = \varnothing$, *then* $\widehat{M} = \varnothing$,
- *if for all* $\alpha \in \mathcal{J}^N_{\leq k}$ *and all* $a \in (\prod_{j:\alpha_j=0} A_j) \setminus \Sigma_\alpha$ *the fiber* $M_{a,\alpha}$ *is thin in* $\prod_{j:\alpha_j=1} D_j$, *then* $\widehat{M}$ *is analytic in* $\widehat{\mathbf{X}}_{N,k}$.

2. Prerequisities

Theorem 2.1 (see [JarPfl 2007], Theorem 1.1)**.** *Let* D_j *be a Riemann domain of holomorphy over* $\mathbb{C}^{n_j}$, $A_j \subset D_j$ *be locally pluriregular and let* $\Sigma_j \subset A'_j \times A''_j$ *be pluripolar,* $j = 1, \dots, N$. *Put*

$$\mathbf{X} := \mathbb{X}((A_j, D_j)_{j=1}^N), \quad \mathbf{T} := \mathbb{T}((A_j, D_j, \Sigma_j)_{j=1}^N).$$

Let $\mathcal{F} \subset \{f : c(\mathbf{T}) \setminus M \to \mathbb{C}\}$ *and let* $M \subset \mathbf{T}$ *be such that:*

- *for any* $j \in \{1, \dots, N\}$ *an any* $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ *the fiber* $M_{(a'_j, \cdot, a''_j)}$ *is pluripolar,*

(4) When $k = N$ we assume that there exists $\widetilde{f} \in \mathcal{O}(D_1 \times \ldots \times D_N \setminus \widetilde{M})$ such that $\widetilde{f} = f$ on $c(\mathbf{W}_{N,k}) \setminus M$.

- *for any* $j \in \{1,\dots,N\}$ *and any* $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ *there exists a closed, pluripolar set* $\widetilde{M}_{a,j} \subset D_j$ *such that* $\widetilde{M}_{a,j} \cap A_j \subset M_{(a'_j,\cdot,a''_j)}$,
- *for any* $a \in c(\mathbf{T}) \setminus M$ *there exists an* $r > 0$ *such that for all* $f \in \mathcal{F}$ *there exists an* $f_a \in \mathcal{O}(\mathbb{P}(a,r))$ *with* $f_a = f$ *on* $\mathbb{P}(a,r) \cap (c(\mathbf{T}) \setminus M)^{(5)}$,
- *for any* $f \in \mathcal{F}$, *any* $j \in \{1,\dots,N\}$, *and any* $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ *there exists a function* $\widetilde{f}_{a,j} \in \mathcal{O}(D_j \setminus \widetilde{M}_{a,j})$ *such that* $\widetilde{f}_{a,j} = f(a'_j, \cdot, a''_j)$ *on* $A_j \setminus M_{a,j}$.

Then there exists a relatively closed, pluripolar set $\widehat{M} \subset \widehat{\mathbf{X}}$ *such that:*

- $\widehat{M} \cap c(\mathbf{T}) \subset M$,
- *for any* $f \in \mathcal{F}$ *there exists a function* $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}} \setminus \widehat{M})$ *such that* $\widehat{f} = f$ *on* $c(\mathbf{T}) \setminus M$,
- $\widehat{M}$ *is singular with respect to* $\{\widehat{f} : f \in \mathcal{F}\}$,
- *if for all* $j \in \{1,\dots,N\}$ *and all* $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ *we have* $\widetilde{M}_{a,j} = \varnothing$, *then* $\widehat{M} = \varnothing$,
- *if for all* $j \in \{1,\dots,N\}$ *and all* $(a'_j, a''_j) \in (A'_j \times A''_j) \setminus \Sigma_j$ *the set* $\widetilde{M}_{a,j}$ *is thin in* D_j, *then* $\widehat{M}$ *is analytic in* $\widehat{\mathbf{X}}$.

Lemma 2.2 ([JarPfl 2010], Lemma 4)**.** *Let* D_j *be a Riemann domain of holomorphy over* $\mathbb{C}^{n_j}$ *and* $A_j \subset D_j$ *be locally pluriregular,* $j = 1,\dots,N$. *Then for all* $z = (z_1,\dots,z_N) \in \widehat{\mathbf{X}}_{N,k}$ *we have:*

$$\mathbf{h}_{\widehat{\mathbf{X}}_{N,k-1},\widehat{\mathbf{X}}_{N,k}}(z) = \max\big\{0, \sum_{j=1}^{N} \mathbf{h}_{A_j,D_j}(z_j) - k + 1\big\}.$$

Theorem 2.3 (Cross theorem for (N,k)-crosses, cf. [JarPfl 2011], Theorem 7.2.7)**.** *Let* D_j *be a Riemann domain of holomorphy over* $\mathbb{C}^{n_j}$ *and* $A_j \subset D_j$ *be locally pluriregular,* $j = 1,\dots,N$. *For* $k \in \{1,\dots,N\}$ *let* $\mathbf{X}_{N,k} := \mathbb{X}_{N,k}((A_j,D_j)_{j=1}^N)$. *Then for every* $f \in \mathcal{O}_{\mathcal{S}}(\mathbf{X}_{N,k})$ *there exists a unique function* $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k})$ *such that* $\widehat{f} = f$ *on* $\mathbf{X}_{N,k}$ *and* $\widehat{f}(\widehat{\mathbf{X}}_{N,k}) \subset f(\mathbf{X}_{N,k})$.

Theorem 2.4 (Cross theorem for generalized (N,k)-crosses)**.** *Let* D_j *be a Riemann domain over* $\mathbb{C}^{n_j}$, $A_j \subset D_j$ *be pluriregular,* $j = 1,\dots,N$. *For* $\alpha \in \mathcal{I}_k^N$ *let* Σ_α *be a subset of* $\prod_{j:\alpha_j=0} A_j$. *Let*

$$\mathbf{X}_{N,k} := \mathbb{X}_{N,k}((A_j,D_j)_{j=1}^N), \quad \mathbf{T}_{N,k} := \mathbb{T}_{N,k}((A_j,D_j)_{j=1}^N, (\Sigma_\alpha)_{\alpha\in\mathcal{I}_k^N}).$$

Then for every $f \in \mathcal{O}_{\mathcal{S}}^c(\mathbf{T}_{N,k})$ *there exists* $\widehat{f} \in \mathcal{O}(\widehat{\mathbf{X}}_{N,k})$ *such that* $\widehat{f} = f$ *on* $\mathbf{T}_{N,k}$ *and* $\widehat{f}(\widehat{\mathbf{X}}_{N,k}) \subset f(\mathbf{T}_{N,k})$.

3. Sketch of proof of Theorem 1.11

Lemma 3.1. *Theorem 1.11 with* $\mathbf{W}_{N,k} = \mathbf{X}_{N,k}$ *implies Theorem 1.11 with*

$$\mathbf{W}_{N,k} \in \{\mathbf{T}_{N,k}, \mathbf{Y}_{N,k}\}.$$

Sketch of proof of Theorem 1.11 with $\mathbf{W}_{N,k} = \mathbf{X}_{N,k}$.
Step 1. Theorem 1.11 is true for any N when $k = 1$ (Theorem 2.1) and when $k = N$ (in this case we assumed the thesis).

(5) $\mathbb{P}(a,r)$ denotes a polydisc in Riemann domain $D_1 \times \ldots \times D_N$ centered at a with radius r.

Step 2. In particular, theorem is true for $N=2$, $k=1,2$. Assume we already have Theorem 1.11 for $(N-1,k)$, where $k\in\{1,\dots,N-1\}$ and for $(N,1)$,…,$(N,k-1)$, where $k\in\{2,\dots,N-1\}$. We need to prove it for (N,k).

Step 3. Fix $s\in\{1,\dots,N\}$ (to simplify the notation let $s=N$). Let

$$Q_N := \{a_N\in A_N : M_{(\cdot,a_N)} \text{ is not pluripolar}\}.$$

Then Q_N is pluripolar. Define

$$\mathbf{X}^{(s)}_{N-1,k} := \mathbb{X}_{N-1,k}((A_j,D_j)_{j=1,j\neq s}^N),\ \ s=1,\dots,N,$$

in particular

$$\mathbf{X}^{(N)}_{N-1,k} = \mathbf{X}_{N-1,k} := \mathbb{X}_{N-1,k}((A_j,D_j)_{j=1}^{N-1}).$$

Fix $a_N\in A_N\setminus Q_N$ and define a family $\{f(\cdot,a_N):\ f\in\mathcal{F}\}\subset\{f:\ c(\mathbf{X}_{N-1,k})\to\mathbb{C}\}$. Then from inductive assumption we get a relatively closed pluripolar set $\widehat{M}_{a_N}\subset\widehat{\mathbf{X}}_{N-1,k}$ such that:

- $\widehat{M}_{a_N}\cap c(\mathbf{X}_{N-1,k})\subset M_{(\cdot,a_N)}$,
- for any $f\in\mathcal{F}$ there exists $\widehat{f}_{a_N}\in\mathcal{O}(\widehat{\mathbf{X}}_{N-1,k}\setminus\widehat{M}_{a_N})$ such that $\widehat{f}_{a_N}=f(\cdot,a_N)$ on $c(\mathbf{X}_{N-1,k})\setminus M_{(\cdot,a_N)}$,
- $\widehat{M}_{a_N}$ is singular with respect to $\{\widehat{f}_{a_N}:\ f\in\mathcal{F}\}$,
- if for all $\alpha'\in\{0,1\}^{N-1}:|\alpha'|\le k$ and all $a'\in(\prod\limits_{j:\alpha'_j=0}A_j)\setminus\Sigma_{\alpha'}$, we have $\widetilde{M}_{a',\alpha'}=\varnothing$, then $\widehat{M}_{a_N}=\varnothing$,
- if for all $\alpha'\in\{0,1\}^{N-1}:|\alpha'|\le k$ and all $a'\in(\prod\limits_{j:\alpha'_j=0}A_j)\setminus\Sigma_{\alpha'}$, the set $\widetilde{M}_{a',\alpha'}$ is thin in $\prod\limits_{j:\alpha'_j=1}D_j$, then $\widehat{M}_{a_N}$ is analytic in $\widehat{\mathbf{X}}_{N-1,k}$.

Define a new cross

$$\mathbf{Z}_N := \mathbb{X}(c(\mathbf{X}_{N-1,k}),A_N;\widehat{\mathbf{X}}_{N-1,k},D_N).$$

Observe that $\mathbf{Z}_N$ with original M, $(\Sigma_\alpha)_\alpha$ and the family $\mathcal{F}$ satisfies all the assumptions of Theorem 1.11 with $N=2$, $k=1$. Then there exists an $\widehat{M}_N\subset\widehat{\mathbf{Z}}_N$, relatively closed, pluripolar, such that:

- $\widehat{M}_N\cap c(\mathbf{X}_{N,k})\subset M$,
- for any $f\in\mathcal{F}$ there exists $\widehat{f}_N\in\mathcal{O}(\widehat{\mathbf{Z}}_N\setminus\widehat{M}_N)$ such that $\widehat{f}_N=f$ on $c(\mathbf{X}_{N,k})\setminus M$,
- $\widehat{M}_N$ is singular with respect to $\{\widehat{f}_N:\ f\in\mathcal{F}\}$,
- if for all $a'\in c(\mathbf{X}_{N-1,k})\setminus\Sigma_{(0,\dots,0,1)}$ we have $\widetilde{M}_{a'}=\varnothing$ and for all $a_N\in A_N\setminus Q_N$ we have $\widetilde{M}_{a_N}=\varnothing$, then $\widehat{M}_N=\varnothing$,
- if for all $a'\in c(\mathbf{X}_{N-1,k})\setminus\Sigma_{(0,\dots,0,1)}$ the set $\widetilde{M}_{a'}$ is thin in D_N and for all $a_N\in A_N\setminus Q_N$ the set $\widetilde{M}_{a_N}$ is thin in $\widehat{\mathbf{X}}_{N-1,k}$, then $\widehat{M}_N$ is analytic in $\widehat{\mathbf{Z}}_N$.

We repeat the reasoning above for all $s=1,\dots,N-1$, obtaining a family of functions $\{\widehat{f}_s\}_{s=1}^N$ such that for any $s\in\{1,\dots,N\}$ we have $\widehat{f}_s=f$ on $c(\mathbf{X}_{N,k})\setminus M$. Define a new function

$$F_f(z) := \begin{cases}\widehat{f}_1(z) & \text{for}\quad z\in\widehat{\mathbf{Z}}_1\setminus\widehat{M}_1\\ \ \vdots & \\ \widehat{f}_N(z) & \text{for}\quad z\in\widehat{\mathbf{Z}}_N\setminus\widehat{M}_N\end{cases}.$$

Lemma 3.2. *Function F_f is well defined and holomorphic on* $\left(\bigcup_{s=1}^{N}\mathbf{Z}_s\right)\setminus\left(\bigcup_{s=1}^{N}\widehat{M}_s\right)$.

Step 4. Define a 2-fold cross

$$\mathbf{Z} := \mathbb{X}(\mathbf{X}_{N-1,k-1}, A_N; \widehat{\mathbf{X}}_{N-1,k}, D_N) \subset \bigcup_{s=1}^{N}\mathbf{Z}_s,$$

a pluripolar set

$$\widetilde{M} := (\bigcup_{s=1}^{N}\widehat{M}_s) \cap (\mathbf{X}_{N-1,k-1}\times A_N)$$

and a family

$$\widetilde{\mathcal{F}} := \{\widetilde{f} := F_f|_{(\mathbf{X}_{N-1,k-1}\times A_N)\setminus\widetilde{M}} : \ f\in\mathcal{F}\}.$$

Then $\mathbf{Z}$, $\widetilde{M}$ and $\widetilde{\mathcal{F}}$ satisfy the assumptions of Theorem 1.11 with $N=1$ and $k=1$. Now from Theorem 1.11 there exists a relatively closed, pluripolar set $\widehat{M}\subset\widehat{\mathbf{Z}}=\widehat{\mathbf{X}}_{N,k}$ such that:

- $\widehat{M}\cap(\mathbf{X}_{N-1,k-1}\times A_N)\subset\widetilde{M}$, in particular, $\widehat{M}\cap c(\mathbf{X}_{N,k})\subset M$,
- for any $f\in\mathcal{F}$ there exists $\widehat{f}\in\mathcal{O}(\widehat{\mathbf{X}}_{N,k}\setminus\widehat{M})$ such that $\widehat{f}=\widetilde{f}$ on $(\mathbf{X}_{N-1,k-1}\times A_N)\setminus\widetilde{M}$, in particular $\widehat{f}=f$ on $c(\mathbf{X}_{N,k})\setminus M$,
- $\widehat{M}$ is singular with respect to $\{\widehat{f}: \ f\in\mathcal{F}\}$,
- if for all $z'\in\mathbf{X}_{N-1,k-1}\setminus P$ $\widetilde{M}_{z'}=\varnothing$ and for all $a_N\in A_N\setminus Q$ $\widetilde{M}_{a_N}=\varnothing$, then $\widehat{M}=\varnothing$,
- if for all $z'\in\mathbf{X}_{N-1,k-1}\setminus P$ the set $\widetilde{M}_{z'}$ is thin in D_N and for all $a_N\in A_N\setminus Q$ the set $\widetilde{M}_{a_N}$ is thin in $\widehat{\mathbf{X}}_{N-1,k}$, then $\widehat{M}$ is analytic in $\widehat{\mathbf{Z}}$.

□

Sketch of proof of Lemma 3.2. Fix s and p. We want to show that $\widehat{f}_s=\widehat{f}_p$ on $(\mathbf{Z}_s\cap\mathbf{Z}_p)\setminus(\widehat{M}_s\cup\widehat{M}_p)$. To simplify the notation we may assume that $s=N-1$ and $p=N$.

Step 1. Every connected component of $\mathbf{Z}_{N-1}\cap\mathbf{Z}_N$ contains part of the center.
Step 2. One connected component of $\widehat{\mathbf{Z}}_{N-1}\cap\widehat{\mathbf{Z}}_N$ contains whole $\mathbf{Z}_{N-1}\cap\mathbf{Z}_N$.
Step 3. Every connected component of $\widehat{\mathbf{Z}}_{N-1}\cap\widehat{\mathbf{Z}}_N$ with $\widehat{M}_{N-1}\cup\widehat{M}_N$ deleted is a domain, thus it is a connected component of $(\widehat{\mathbf{Z}}_{N-1}\cap\widehat{\mathbf{Z}}_N)\setminus(\widehat{M}_{N-1}\cup\widehat{M}_N)$.
Step 4. One connected component of $(\widehat{\mathbf{Z}}_{N-1}\cap\widehat{\mathbf{Z}}_N)\setminus(\widehat{M}_{N-1}\cup\widehat{M}_N)$ contains whole set $(\mathbf{Z}_{N-1}\cap\mathbf{Z}_N)\setminus(\widehat{M}_{N-1}\cup\widehat{M}_N)$.
Step 5. $\widehat{f}_{N-1}=\widehat{f}_N$ on $(\mathbf{Z}_{N-1}\cap\mathbf{Z}_N)\setminus(\widehat{M}_{N-1}\cup\widehat{M}_N)$. □

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