Variational method and extremal mappings

Let $\beta \in (0,1)$ and $\omega \in \mathcal{A}(A_\beta, \mathbb{C}^n)$, where $A_\beta := \{\lambda : \beta < |\lambda| < 1\}$ and $\mathcal{A}(A_\beta, \mathbb{C}^n) := \mathcal{O}(A_\beta, \mathbb{C}^n) \cap \mathcal{C}(\overline{A_\beta}, \mathbb{C}^n)$. We define a linear functional $\Phi : H^\infty(\mathbb{D}, \mathbb{C}^n) \to \mathbb{R}$ by

$$\Phi(h) := \frac{1}{2\pi} \int_0^{2\pi} \mathfrak{Re} \left(h^*(e^{i\theta}) \bullet \omega(e^{i\theta}) \right) d\theta, \tag{1}$$

where $h^*(e^{i\theta})$ is the non-tangential boundary value of h at $e^{i\theta} \in \mathbb{T}$.

Definition 1 (Problem (P))**.** Let $N \in \mathbb{N}$ and $\Phi_1, ..., \Phi_N$ are functionals of the form (1). Suppose that we are given a bounded domain $D \subset \mathbb{C}^n$ and real numbers $a_1, ..., a_N$. We want to find a mapping $f \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n)$ such that $\Phi_j(f) = a_j$ for $j = 1, ..., N$ and there is no mapping $g \in \mathcal{O}(\mathbb{D}, \mathbb{C}^n)$ such that $\Phi_j(g) = a_j$, $j = 1, ..., N$ and $g(\mathbb{D}) \Subset D$. Any solution of (P) is called an *extremal mapping* for (P).

Definition 2. We shall call a function $p : L^1(\mathbb{T}, \mathbb{C}^n) \to [0, +\infty)$ the *Minkowski pseudonorm*, if

- $p(h_1 + h_2) \le p(h_1) + p(h_2)$ for any $h_1, h_2 \in L^1(\mathbb{T}, \mathbb{C}^n)$,
- $p(th) = tp(h)$ for any $t \ge 0$ and $h \in L^1(\mathbb{T}, \mathbb{C}^n)$,
- there exists $M \in (0, +\infty)$ such that $p(h) \le M|h|_1$, where $|\cdot|_1$ is a norm in $L^1(\mathbb{T}, \mathbb{C}^n)$.

The main result of the talks is the following:

Theorem 3. *Let $f : \mathbb{D} \to \mathbb{C}^n$ be a bounded holomorphic mapping and let G be a bounded, open and connected neighborhood of $\overline{f(\mathbb{D})}$. Let $u \in PSH(D) \cap \mathcal{C}(D)$ be such that $u(f) < 0$ on $\mathbb{D}$ and moreover let $p : L^1(\mathbb{T}, \mathbb{C}^n) \to [0, +\infty)$ be the Minkowski pseudonorm with*

- *$\int_0^{2\pi} [u(f^*(e^{i\theta}) + h(e^{i\theta}))]^+ d\theta \le p(h) + o(||h||_\infty)$ for all $h \in \mathcal{A}(\mathbb{D}, \mathbb{C}^n)$ such that $f + h : \mathbb{D} \to D$.*

Assume that f is an extremal for $(\mathbf{P})$ with data $(\Phi_1, \Phi_1(f)), ..., (\Phi_N, \Phi_N(f))$ in $\{z \in D : u(z) < 0\}$. Then, there are: $\lambda_1, ..., \lambda_N \in \mathbb{R}$, $\sum_{l=1}^N |\lambda_l| > 0$, and $g \in H^\infty(\mathbb{D}, \mathbb{C}^n)$, $g(0) = 0$ such that

$$\int_0^{2\pi} \mathfrak{Re} \left(\left(\sum_{k=1}^N \lambda_k \omega_k(e^{i\theta}) + g^*(e^{i\theta}) \right) \bullet h(e^{i\theta}) \right) d\theta \le p(\mathbf{1_B} \cdot h) \tag{2}$$

for all $h \in L^1(\mathbb{T}, \mathbb{C}^n)$, where $\mathbf{B}$ is a set of all points $\theta \in [0, 2\pi)$ such that $f^*(e^{i\theta})$ exists and $u(f^*(e^{i\theta})) = 0$, $\mathbf{1_B}$ is a characteristic function for $\mathbf{A}$.

As a application of above theorem we will give a characterization of all possible extremal mappings in the sense of Lempert function and in the sense of Kobayashi-Royden pseudometric for a large class of domains in $\mathbb{C}^2$.