LEMPERT THEOREM

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In 1984 L. Lempert [4] published the proof of the equality of the Lempert function and the Carathéodory distance on strictly linearly convex domains with analytic boundaries [4], however the proof contained some gaps. Our goal is to repair the fragments which were not correctly done during the conference Geometric Methods in Complex Analysis II, Będlewo 2011.

Theorem 1 (Lempert Theorem)**.** *Let $D \subset \mathbb{C}^n$ be a strictly linearly convex domain with an analytic boundary. Then*

$$c_D = k_D = \widetilde{k}_D \text{ and } \gamma_D = \kappa_D.$$

It is possible to describe all the $\widetilde{k}_D$- and κ_D-extremals.

Definition 2. *Let $D \subset \mathbb{C}^n$ be a strictly linearly convex domain with an analytic boundary. We call a holomorphic mapping $f : \mathbb{D} \longrightarrow D$ an E-*mapping *if*

(1) f extends to a $\mathcal{C}^\omega$ function on $\overline{\mathbb{D}}$ (denoted by the same letter f);
(2) $f(\partial\mathbb{D}) \subset \partial D$;
(3) there exists a positive $\mathcal{C}^\omega$ function $\rho : \partial\mathbb{D} \longrightarrow \mathbb{R}$ such that the mapping $\partial\mathbb{D} \ni \zeta \longmapsto \zeta\rho(\zeta)\overline{\nu(f(\zeta))} \in \mathbb{C}^n$ extends to a $\mathcal{C}^\omega$ function $\widetilde{f} : \overline{\mathbb{D}} \longrightarrow \mathbb{C}^n$, holomorphic in $\mathbb{D}$, where $\nu(z)$ is the outward unit normal vector to ∂D at z;
(4) setting $\varphi(\zeta) := \overline{\nu(f(\zeta))}(z - f(\zeta))$, $\zeta \in \partial\mathbb{D}$, we have $\operatorname{wind}\varphi = 0$ for all $z \in D$.

Theorem 3. *Let D be a strictly linearly convex domain with an analytic boundary. Then for any different points $z, w \in D$ (resp. $z \in D$, $v \in (\mathbb{C}^n)_*$) there exists an E-mapping $f : \mathbb{D} \longrightarrow D$ such that $f(0) = z$, $f(\xi) = w$ for some $\xi \in (0,1)$ (resp. $f(0) = z$, $f'(0) = \lambda v$ for some $\lambda > 0$). Moreover, f is the unique $\widetilde{k}_D$-extremal w.r.t. z, w (resp. the unique κ_D-extremal w.r.t. z, v).*

References

[1] J. E. Fornæss, *Embedding strictly pseudoconvex domains in convex domains*, Am. J. Math. **98** (1976), 529–569.
[2] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Nauka, Moscow, 1966.
[3] M. Jarnicki, P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, Walter de Gruyter, 1993.
[4] L. Lempert, *Intrinsic distances and holomorphic retracts*, in *Complex analysis and applications '81 (Varna, 1981)*, 341–364, Publ. House Bulgar. Acad. Sci., Sofia, 1984.
[5] —, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France, **109** (1981), 427–474.
[6] W. Rudin, *Function Theory in the Unit Ball of $\mathbb{C}^n$*, Springer-Verlag, Berlin, 2008.
[7] E. Tadmor, *Complex symmetric matrices with strongly stable iterates*, Linear algebra and its applications **78** (1986), 65–77.