Plurisubharmonic subextensions as envelopes on disc functionals

(based on the paper by F. Lárusson and E. Poletsky)

Let $W \subset X \subset \mathbb{C}^N$ be domains, and let $\varphi : W \to [-\infty, \infty)$ be upper semicontinuous.

Definition 1.

- $\mathcal{A}_X := \mathcal{O}(\mathbb{D}, X) \cap \mathbb{C}(\overline{\mathbb{D}}, X)$, endowed with the topology of the supremum norm on $\overline{\mathbb{D}}$,
- $\mathcal{A}_X^W := \{f \in \mathcal{A}_X : f(\mathbb{T}) \subset W\}$,
- $S_W\varphi(x) := \sup\{u(x) : u \in PSH(X), u|_W \leq \varphi\}$, $x \in X$,
- for a family $\mathcal{B} \subset \mathcal{A}_X^W$:

$$E_{\mathcal{B}}\varphi(x) := \inf_{f \in \mathcal{B}, f(0)=x} \int_{\mathbb{T}} \varphi \circ f d\sigma,\ x \in X,$$

  where σ denotes the normalized Lebesgue measure on $\mathbb{T}$.

Observation 2. *$S_W\varphi \leq E_{\mathcal{A}_X^W}\varphi$ on X.*

Remark 3. It is a classical result that $S_X\varphi = E_{\mathcal{A}_X}\varphi$ on X.

Definition 4. We say that discs $f_0, f_1 \in \mathcal{A}_X^W$ are centre-homotopic, if $f_0(0) = f_1(0)$ and there is a continuous mapping $F : [0, 1] \to \mathcal{A}_X^W$ such that $F(0) = f_0, F(1) = f_1$ and $F(t)(0) = f_0(0)$ for each t.

Definition 5. A family $\beta = (\beta_\tau : U_\tau \to \mathcal{A}_X^W)_{\tau \in T}$ of continuous mappings is called a W-disc structure on X, if:

- $(U_\tau)_{\tau \in T}$ is an open cover of X,
- $\beta_\tau(x)(0) = x$ for each $\tau \in T, x \in U_\tau$,
- $\beta_{\tau_1}(x)$ and $\beta_{\tau_2}(x)$ are centre-homotopic for each $x \in U_{\tau_1} \cap U_{\tau_2}$, $\tau_1, \tau_2 \in T$.

For a W-disc structure β on X we define $\mathcal{B}_\beta := \bigcup_{\tau \in T} \beta_\tau(U_\tau)$ and $E_\beta\varphi := E_{\mathcal{B}_\beta}\varphi$.

We say that X is a *schlicht disc extension* of W if there exists a W-disc structure β on X with the following property:

there is $\tau \in T$ such that $U_\tau = W$ and $\beta_\tau(w)$ is the constant disc at w for every $w \in W$.

Theorem 6. *Let $W \subset X \subset \mathbb{C}^N$ be domains such that X is a schlicht disc extension of W. If $\varphi : W \to [-\infty, \infty)$ is upper semicontinuous, then*

$$S_W\varphi = E_{\mathcal{A}_X^W}\varphi.$$

Theorem 7. *Let $W \subset X \subset \mathbb{C}^N$ be domains. Assume that there is a connected component $\mathcal{B}$ of $\mathcal{A}_X^W$ such that:*

- *$\mathcal{B}$ covers X, i.e. $\bigcup_{f \in \mathcal{B}} f(0) = X$,*
- *every $f_0, f_1 \in \mathcal{B}$ satisfying $f_0(0) = f_1(0)$ are centre-homotopic.*

If $\varphi : W \to [-\infty, \infty)$ is upper semicontinuous, then

$$S_W\varphi = E_{\mathcal{A}_X^W}\varphi.$$

Lemma 8. *Let $W \subset X \subset \mathbb{C}^N$ be domains, let $\varphi : W \to [-\infty, \infty)$ be upper semicontinuous, and let β be a W-disc structure on X. Then $E_\beta\varphi$ is upper semicontinuous on X and*

$$E_{\mathcal{A}_X^W}\varphi \leq E_{\mathcal{A}_X}E_\beta\varphi.$$