ESTIMATES OF INVARIANT FUNCTIONS ON CONVEX AND $\mathbb{C}$-CONVEX DOMAINS

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The following propositions due to N. Nikolov are proved (d_D, c_D, k_D, b_D, $\widetilde{k}_D$ are respectively the distance to ∂D, the Carathéodory, Kobayashi and Bergman distances and the Lempert function of D).

Proposition 1. *Let $D \subsetneq \mathbb{C}^n$ be a convex domain. Then*

$$\widetilde{k}_D(z,w) \le \begin{cases} \frac{|z-w|}{d_D(z)-d_D(w)} \log \frac{d_D(z)}{d_D(w)}, & z,w \in D,\ d_D(z) \neq d_D(w), \\ \frac{|z-w|}{d_D(z)}, & z,w \in D,\ d_D(z) = d_D(w). \end{cases}$$

In particular, if D is bounded, then for any compact set $K \subset D$ there is $c_K > 0$ such that

$$b_D(z,w) \le -c_K \log d_D(w) + c_K, \quad z \in K,\ w \in D.$$

Proposition 2. *Let $D \subsetneq \mathbb{C}^n$ be a $\mathbb{C}$-convex domain. Then*

$$c_D(z,w) \ge \frac{1}{4} \log \frac{d_D(z)}{4 d_D(w)}, \quad z,w \in D.$$

In particular, if D is bounded then for any compact set $K \subset D$ there is $c_K > 0$ such that

$$b_D(z,w) \ge -\frac{1}{4} \log d_D(w) - c_K, \quad z \in K,\ w \in D.$$

Proposition 3. *Let $D \subset \mathbb{C}^n$ be a bounded domain and let s_D be k_D or b_D.*

(*a*) *If D is locally $\mathbb{C}$-convexifiable then there exists $c > 0$ such that for any compact set $K \subset D$ there is $c_K > 0$ satisfying*

$$s_D(z,w) \ge -c \log d_D(w) - c_K, \quad z \in K,\ w \in D.$$

(*b*) *If D is locally $\mathbb{C}$-convexifiable and $\mathcal{C}^{1+\varepsilon}$-smooth then there exists $c > 0$ such that for any compact set $K \subset D$ there is $c_K > 0$ satisfying*

$$s_D(z,w) \le -c \log d_D(w) + c_K, \quad z \in K,\ w \in D.$$

(*c*) *If D is locally convexifiable then for any compact set $K \subset D$ there is $c_K > 0$ such that*

$$s_D(z,w) \le -c_K \log d_D(w) + c_K, \quad z \in K,\ w \in D.$$

Proposition 4. *Let p be a $\mathcal{C}^{1+\varepsilon}$-smooth boundary point of a domain $D \subset \mathbb{C}$ and let s_D be c_D, k_D or $b_D/\sqrt{2}$. Then*

(*a*) *For any sufficiently small neighborhood U of p there exist a neighborhood V_U of p and a constant $c_U > 0$ such that*

$$s_D(z,w) \ge -\frac{1}{2} \log d_D(w) - c_U, \quad z \in D \setminus U,\ w \in D \cap V_U.$$

(*b*) *For any compact set $K \subset D$ there exist a neighborhood V_K of p and a constant $c_K > 0$ such that*

$$s_D(z,w) \le -\frac{1}{2} \log d_D(w) + c_K, \quad z \in K,\ w \in D \cap V_K.$$

References

[1] N. Nikolov, *Estimates of invariant distances on "convex" domains*, arXiv:1210.7223.