Complex geodesics in convex tube domains

Definition 1. We say that a domain $D \subset \mathbb{C}^n$ is a convex tube, if $D = \Omega + i\mathbb{R}^n$ for some convex domain $\Omega \subset \mathbb{R}^n$. We call Ω the *base* of D, and denote it by $\operatorname{Re} D$. We say that D is bounded from the right side, if there is $c \in \mathbb{R}$ such that $\operatorname{Re} D \subset (-\infty, c)^n$.

Theorem 2. *Let $D \subset \mathbb{C}^n$ be a convex tube domain, bounded from the right side, and let $\varphi : \mathbb{D} \to D$ be a holomorphic map. Then φ is a complex geodesic for D iff there exists a mapping $h : \mathbb{C} \to \mathbb{C}^n$ of the form $\bar{a}\lambda^2 + b\lambda + a$ with some $a \in \mathbb{C}^n, b \in \mathbb{R}^n$, such that:*

(i) $\operatorname{Re} \left[\bar{\lambda} h(\lambda) \bullet (z - \varphi^*(\lambda))\right] < 0$ *for all $z \in D$ and a.e. $\lambda \in \mathbb{T}$,*

(ii) $\operatorname{Re} \left[h(\lambda) \bullet \frac{\varphi(0)-\varphi(\lambda)}{\lambda}\right] < 0$ *for every $\lambda \in \mathbb{D}_*$.*

Moreover, if φ is a complex geodesic for D, then h may be chosen as

$$h(\lambda) := \left(\frac{\partial f}{\partial z_1}(\varphi(\lambda)), \dots, \frac{\partial f}{\partial z_n}(\varphi(\lambda))\right), \ \lambda \in \mathbb{D},$$

where f is a left inverse for φ (h is then of the required form $\bar{a}\lambda^2 + b\lambda + a$, $a \in \mathbb{C}^n$, $b \in \mathbb{R}^n$).

Lemma 3. *Let $D \subset \mathbb{C}^n$ be a convex tube, bounded from the right side, let $\varphi : \mathbb{D} \to D$ be a holomorphic map with the boundary measure μ*[1]*, and let $h(\lambda) = \bar{a}\lambda^2 + b\lambda + a$, $\lambda \in \mathbb{D}$, for some $a \in \mathbb{C}^n$, $b \in \mathbb{R}^n$, with $h \not\equiv 0$. Then*

(m) *the measure $\bar{\lambda} h(\lambda) \bullet (\operatorname{Re} z \, d\mathcal{L}^{\mathbb{T}}(\lambda) - d\mu(\lambda))$ is negative for every $z \in D$*

iff the following two conditions holds:

(i) $\operatorname{Re} \left[\bar{\lambda} h(\lambda) \bullet (z - \varphi^*(\lambda))\right] < 0$ *for all $z \in D$ and a.e. $\lambda \in \mathbb{T}$,*

(ii) $\operatorname{Re} \left[h(\lambda) \bullet \frac{\varphi(0)-\varphi(\lambda)}{\lambda}\right] < 0$ *for every $\lambda \in \mathbb{D}_*$.*

Theorem 4. *Let $D \subset \mathbb{C}^n$ be a convex tube, bounded from the right side, and let $\varphi : \mathbb{D} \to D$ be a holomorphic map with the boundary measure μ. Then φ is a complex geodesic for D iff there exists a mapping $h : \mathbb{C} \to \mathbb{C}^n$ of the form $\bar{a}\lambda^2 + b\lambda + a$ with $a \in \mathbb{C}^n$, $b \in \mathbb{R}^n$, $h \not\equiv 0$, such that*

the measure $\bar{\lambda} h(\lambda) \bullet (\operatorname{Re} z \, d\mathcal{L}^{\mathbb{T}}(\lambda) - d\mu(\lambda))$ is negative for every $z \in D$.

[1] That is, μ is the only n-tuple of real, finite, Borel measures on $\mathbb{T}$ s.t.

$$\operatorname{Re} \varphi(\lambda) = \frac{1}{2\pi} \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|\zeta - \lambda|^2} d\mu(\zeta), \ \lambda \in \mathbb{D}.$$