# Lectures on open complex and real analytic maps. A summary

JJ Loeb

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The following theorem is classical in complex analysis.

**Theorem 1** *Let $f$ be an holomorphic map from a domain $U$ of $\mathbb{C}^n$ into a domain $V$ in $\mathbb{C}^n$.*
*There is an equivalence between:*
*1. The fibers of $f$ are discrete.*
*2. The map $f$ is open ( this means that the image of an open set is open).*

The following theorem gives a class of such holomorphic open maps.

**Theorem 2** *A proper holomorphic map $f$ from a domain $U$ of $\mathbb{C}^n$ into a domain $V$ in $\mathbb{C}^n$ is open.*

As a corollary, we see that such a map is onto.

Examples:
1. A nonconstant holomorphic polynomial in one variable is onto (fundamental theorem of algebra).
2. If $n = 1$, a nonconstant holomorphic map is open.
3. A polynomial map of the form $\sum_0^l P_k(z)$ with $P_k$ homogeneous of degree $k$ and such that $P_l^{-1}(0) = 0$ is proper and then is also open.
4. An example of a nonopen polynomial map is $(x, y) \to (xy, y)$. Here the Jacobian does'nt vanish identically.

Sketch of proof ( 1 implies 2) in theorem 1: Using a topological lemma and the discreteness of the fibers, one sees that for every point $p$ in $U$, there exists a neighberhood $U_1$ of $p$ and a neighberhood $V_1$ of $f(p)$ such that $f$ is a proper map from $U_1$ into $V_1$. Using the well known fact that the Jacobian is nonnegative for holomorphic maps, one can deduce that the Brouwer degree is stricly positive. Then $f$ is onto. It is easy to deduce that $f$ is open.

The proof of theorem 2 uses the fact that a compact complex analytic set in $\mathbb{C}^n$ is finite.

We can generalize theorem 1. in the following direction (see Gamboa-Ronga for the polynomial case and Hirsh for the real analytic case).

**Theorem 3** *Let $f$ be a real-analytic map from a domain $U$ of $\mathbb{R}^n$ into a domain $V$ of $\mathbb{R}^n$. Then there is an equivalence between:*

*1. The Jacobian of $f$ has constant sign (This means that it cannot be strictly positive at one point and strictly negative at another) and the fibers of $f$ are discrete.*

*2. $f$ is an open map.*

**remarks**

1. The polynomial map $(x, y) \to (x^2+y^2, 0)$ is proper and the sign of the Jacobian does'nt change. But this map is not open.

2. It seems hard to find open polynomial maps from $\mathbb{R}^2$ into itself and which are not proper. An example is the Pincuck map. The Jacobian of this map is strictly positive at every point.

Sketch of proof of theorem 3: The proof of 1 implies 2 is the same as for theorem 1. A main difficulty in the real case for the proof of 2 implies 1 is that the critical set can contain hypersurfaces. In this case, there exists smooth points such that there is transversality between the tangent space and the kernel of the differential of $f$. At such a point, up to change of charts, the form of $f$ is: $(X, t) \to (X, g(X, t))$. Elementary arguments permit to conclude.

Here is another type of theorem in the real case, which was given by Sheil Small with another proof.

**Theorem 4** *Let $P(z, \overline{z})$ a polynomial map from $\mathbb{C}$ into itself of the form $\sum_0^l P_k(z)$ where the $P_k$ are real homogeneous of degree $k$ and $P_l(z) = z^l$. Then:*

*1. $P$ is into.*

*2. The fibers of $P$ are finite.*

Sketch of proof: The proof of 1. is by using the invariance of Brouwer degree by homotopy. The degree of $P$ is $l > 0$ and then $P$ is onto. The proof of 2. is by using the complexification of $P$ which is a holomorphic map from $\mathbb{C}^2$ into itself given by: $(u, v) \to (P(u, v), \overline{P(\overline{v}, \overline{u})})$. One can see that this complexification is a proper map and then the fibers are finite. Then the fibers of $P$ itself are finite.

Note that in general such maps $P$ are not open.