THE CARATHÉODORY AND KOBAYASHI–ROYDEN METRICS BY WAY OF DUAL EXTREMAL PROBLEMS

HALSEY ROYDEN, PIT-MANN WONG AND STEVEN G. KRANTZ

Let $(X, \| \|_X)$ be a complex normed space and let $P : X \to \mathbb{R}_+$ be the *Minkowski subnorm on* $(X, \| \|_X)$, i.e.

- $P(x + y) \le P(x) + P(y)$, $x, y \in X$,
- $P(tx) = tP(x)$, $t \ge 0$, $x \in X$,
- $c^{-1}\|x\|_X \le P(x) \le c\|x\|_X$, $x \in X$, where $c = c(P, \| \|_X) > 0$ is some constant.

Let

$$X^* := \{u : X \to \mathbb{C} : u \text{ is } \mathbb{C}\text{-linear and continuous}\}$$

denote the dual of X endowed with the standard norm

$$\|u\|_{X^*} := \sup\left\{\frac{|u(x)|}{\|x\|_X} : x \in X \setminus \{0\}\right\}, \quad u \in X^*.$$

Put

$$P^*(u) := \sup\left\{\frac{\operatorname{Re} u(x)}{P(x)} : x \in X \setminus \{0\}\right\}, \quad u \in X^*.$$

One can easily prove that $P^* : X^* \to \mathbb{R}_+$ is a Minkowski subnorm on $(X^*, \| \|_{X^*})$ with the same constant c as that for P.

- Indeed, subadditivity and positive homogeneity are clear. To see the third condition observe that
$$\frac{\operatorname{Re} u(x)}{P(x)} \le c\frac{|u(x)|}{\|x\|_X}, \quad x \ne 0,$$
i.e. right inequality is trivial. To see the left inequality fix $u \in X^*$ and observe that for any $x \in X$ with $u(x) \ne 0$ there is $\tilde{x} \in X$ with $\|x\|_X = \|\tilde{x}\|_X$ and
$$0 < u(\tilde{x}) = |u(\tilde{x})| = \operatorname{Re} u(\tilde{x}).$$
Hence
$$\frac{|u(x)|}{c\|x\|_X} = \frac{\operatorname{Re} u(\tilde{x})}{c\|\tilde{x}\|_X} \le \frac{\operatorname{Re} u(\tilde{x})}{P(\tilde{x})} \le P^*(u).$$
Since x was arbitrary (if $u(x) = 0$ then inequality is obvious), we get
$$c^{-1}\|u\|_{X^*} \le P^*(u).$$

Let Y be a complex linear subspace of X and let $x_0 \in X \setminus \overline{Y}$. Define

$$m = m(x_0 + Y) := \inf_{y \in Y} P(x_0 + y)$$

and

$$M = M(x_0 + Y) := \inf\{P^*(u) : u \in Y^0,\ \operatorname{Re} u(x_0) = 1\},$$

where $Y^0 := \{u \in X^* : u|_Y = 0\}$ is the *annihilator* of Y. Clearly $m > 0$, since $x_0 \notin \overline{Y}$.

To solve the linear extremal problem is to find a point $x \in x_0 + Y$ such that $P(x) = m$. *To solve the dual extremal problem* is to find a point $u \in Y^0$ with $\operatorname{Re} u(x_0) = 1$ such that $P^*(u) = M$.

Proposition 1 (principle of duality)**.** *With notation as above, we have*

(i) *$mM = 1$;*
(ii) *there exists a point $u \in Y^0$ with $\operatorname{Re} u(x_0) = 1$ and $P^*(u) = M$;*
(iii) *if $x \in x_0 + Y$ and $u \in Y^0$ are such that $\operatorname{Re} u(x) = P(x)P^*(u) = 1$, then $P(x) = m$ and $P^*(u) = M$.*

0.1. **A linear extremal problem.** Let Ω be a bounded, convex domain in $\mathbb{C}^n$ containing the origin. Let

$$p(z) := \inf\{\lambda > 0 : \lambda^{-1} z \in \Omega\}, \quad z \in \mathbb{C}^n,$$

be the *Minkowski function* of Ω. Note that $\Omega = \{z \in \mathbb{C}^n : p(z) < 1\}$ and that p is Minkowski subnorm on $\mathbb{C}^n$.

- Indeed, recall that $p_{\Omega_1} \geq p_{\Omega_2}$ for $\Omega_1 \subset \Omega_2$. Form the assumption there are $r, R > 0$ such that $B(0, r) \subset \Omega \subset B(0, R)$, i.e.

$$\frac{\|z\|}{R} = p_{B(0,R)}(z) \leq p(z) \leq p_{B(0,r)}(z) = \frac{\|z\|}{r}, \quad z \in \mathbb{C}^n.$$

  It suffices to take $c := \max\{r^{-1}, R\}$.

For $\nu \in \mathbb{N}$ and $\zeta_\alpha \in \mathbb{D}$, $d_\alpha \in \mathbb{N}$, $1 \leq \alpha \leq \nu$, let

$$\mathcal{D}(\zeta) = \mathcal{D}_{(\zeta_1,\ldots,\zeta_\nu;d_1,\ldots,d_\nu)}(\zeta) := \prod_{\alpha=1}^{\nu} (\zeta - \zeta_\alpha)^{d_\alpha}, \quad \zeta \in \mathbb{C},$$

be the *divisor* on $\mathbb{D}$ with total degree $\deg \mathcal{D} := \sum_{\alpha=1}^{\nu} d_\alpha$ and let

$$\mathfrak{D} := \{a_{\alpha,\beta_\alpha} \in \mathbb{C}^n : 1 \leq \alpha \leq \nu,\ 0 \leq \beta_\alpha \leq d_\alpha - 1\}.$$

For $k \in \mathbb{Z}_+$ put $X := \mathcal{O}(\mathbb{D}, \mathbb{C}^n) \cap \mathcal{C}^k(\overline{\mathbb{D}}, \mathbb{C}^n)$ and

$$\|f\|_X := \sum_{j=0}^{k} \max_{\zeta \in \overline{\mathbb{D}}} \|f^{(j)}(\zeta)\|, \quad f \in X.$$

Then $(X, \| \|_X)$ is a Banach space. Define

$$P(f) := \max_{\zeta \in \overline{\mathbb{D}}} p(f(\zeta)), \quad f \in X.$$

Finally, we define

$$(1.1) \quad L_k = L_k(\mathcal{D}, \mathfrak{D}) := \{f \in X : f^{(\beta_\alpha)}(\zeta_\alpha) = a_{\alpha,\beta_\alpha},\ 1 \leq \alpha \leq \nu,\ 0 \leq \beta_\alpha \leq d_\alpha - 1\}.$$

Note that L_k is an affine subspace of X. Indeed, if we fix $f_0 \in L_k$ then

$$L_k = f_0 + Y,$$

where $Y := \{\mathcal{D}f : f \in X\}$ is a closed linear subspace of X.

Proposition 2. *Let Ω be a bounded, convex domain in $\mathbb{C}^n$ containing the origin and let $k \in \mathbb{Z}_+$.*

(a) *Let $\zeta_1, \zeta_2 \in \mathbb{D}$, $a, b \in \Omega$, $a \neq b$, $\mathcal{D}(\zeta) = (\zeta - \zeta_1)(\zeta - \zeta_2)$, $\mathfrak{D} = \{a, b\}$, $f \in L_k(\mathcal{D}, \mathfrak{D})$ with $f(\mathbb{D}) \subset \Omega$. Then the following are equivalent*
  (i) *f is extremal for $K_\Omega(a, b)$;*
  (ii) *$P(f) = 1$ and f is extremal for $m(L_k)$.*

(b) *Let $a \in \Omega$, $v \in \mathbb{C}^n \setminus \{0\}$, $\mathcal{D}(\zeta) = \zeta^2$, $\mathfrak{D} = \{a, v\}$, $f \in L_k(\mathcal{D}, \mathfrak{D})$ with $f(\mathbb{D}) \subset \Omega$. Then the following are equivalent*
  (i) *f is extremal for $\kappa_\Omega(a; v)$;*
  (ii) *$P(f) = 1$ and f is extremal for $m(L_k)$.*

0.2. **The dual extremal problem.** Let Ω be a bounded, convex domain in $\mathbb{C}^n$ containing the origin with Minkowski function p. Let

$$p^*(w) := \sup_{z\in\mathbb{C}^n\setminus\{0\}} \frac{\operatorname{Re}(z\bullet w)}{p(z)}, \quad w\in\Omega,$$

where $z\bullet w := \sum_{j=1}^n z_j w_j$ for any $z=(z_1,\dots,z_n), w=(w_1,\dots,w_n)\in\mathbb{C}^n$. Let $\mathbb{T}:=\partial\mathbb{D}$.

For $k\in\mathbb{Z}_+$ put $X:=\mathcal{C}^k(\mathbb{T},\mathbb{C}^n)$ and

$$\|f\|_X := \sum_{j=0}^k \max_{\zeta\in\mathbb{T}} \|f^{(j)}(\zeta)\|, \quad f\in X.$$

Then $(X,\|\ \|_X)$ is a Banach space. Define

$$P(f) := \max_{\zeta\in\mathbb{T}} p(f(\zeta)), \quad f\in X.$$

Let P^* be the Minkowski subnorm associated to P defined on the dual space X^*. Denote by τ the normalized Lebesgue measure on $\mathbb{T}$, i.e.

$$\int_{\mathbb{T}} v(\zeta)\,d\tau(\zeta) := \frac{1}{2\pi}\int_0^{2\pi} v(e^{i\theta})\,d\theta, \quad v\in L^1(\mathbb{T},\mathbb{C};\tau).$$

For $v=(v_1,\dots,v_n)\in L^1(\mathbb{T},\mathbb{C}^n;\tau)$ let $v\tau:=(v_1\tau,\dots,v_n\tau)$.

Theorem 3. *Let Ω be a bounded, convex domain in $\mathbb{C}^n$ with $\mathcal{C}^k$ boundary, $k\in\mathbb{Z}_+$, containing the origin and let $f\in L_k(\mathcal{D},\mathfrak{D})$, where $\mathfrak{D}$ contains at least two different vectors. Then the following conditions are equivalent*

(i) f is extremal for $m=m(L_k)$;

(ii) there is a map $\tilde h\in H^1(\mathbb{D},\mathbb{C}^n)$, $h\not\equiv 0$, such that

$$\operatorname{Re}\int_{\mathbb{T}} f(\zeta)\bullet h(\zeta)\,d\tau(\zeta) = P(f)P^*(h\tau),$$

where $h(\zeta):=\zeta(\mathcal{D}(\zeta))^{-1}\tilde h^*(\zeta)$;

(iii) there is a map $\tilde h\in H^1(\mathbb{D},\mathbb{C}^n)$, $h\not\equiv 0$, such that

$$p(f(\zeta))=m, \quad \operatorname{Re}(f(\zeta)\bullet h(\zeta)) = p(f(\zeta))p^*(h(\zeta)) \quad \text{for a.a. } \zeta\in\mathbb{T},$$

where $h(\zeta):=\zeta(\mathcal{D}(\zeta))^{-1}\tilde h^*(\zeta)$.

Furthermore,

- *if Ω is strictly convex (i.e. if $z,w,\frac12(z+w)\in\partial\Omega$, then $z=w$) then f as in (i) is unique;*
- *if Ω has $\mathcal{C}^1$ boundary, then h as in (ii) or (iii) is unique.*

Corollary 4. *Let Ω be a bounded, convex domain in $\mathbb{C}^n$ containing the origin and let $f\in L_k(\mathcal{D},\mathfrak{D})\cap L_k(\mathcal{D}',\mathfrak{D}')$, where $\mathcal{D}$ and $\mathcal{D}'$ are divisors on $\mathbb{D}$ such that $d:=\deg\mathcal{D}\le\deg\mathcal{D}'=:d'$. If f is extremal for $m=m(L_k(\mathcal{D},\mathfrak{D}))$, then it is also extremal for $m'=m'(L_k(\mathcal{D}',\mathfrak{D}'))$.*

Corollary 5. *Let Ω be a bounded, convex domain in $\mathbb{C}^n$ containing the origin and let $k\in\mathbb{Z}_+$. Let $\zeta_1,\zeta_2\in\mathbb{D}$, $a,b\in\Omega$, $a\neq b$, $\mathcal{D}(\zeta)=(\zeta-\zeta_1)(\zeta-\zeta_2)$, $\mathfrak{D}=\{a,b\}$, $f\in L_k(\mathcal{D},\mathfrak{D})$ with $f(\mathbb{D})\subset\Omega$. Assume that f is extremal for $K_\Omega(a,b)$. Then f is also extremal for $m=m(L_k(\mathcal{D},\mathfrak{D}))$ with $P(f)=1$ and there is a map $\tilde h\in H^1(\mathbb{D},\mathbb{C}^n)$, $h\not\equiv 0$, such that for a.a. $\zeta\in\mathbb{T}$ the hyperplane*

$$\{z\in\mathbb{C}^n : \operatorname{Re}((z-f(\zeta))\bullet h(\zeta))=0\}$$

is a supporting hyperplane for Ω at $f(\zeta)$, where $h(\zeta):=\zeta(\mathcal{D}(\zeta))^{-1}\tilde h^*(\zeta)$. Furthermore,

- *if Ω is strictly convex then f is unique;*

- *if* Ω *has* $\mathcal{C}^1$ *boundary, then* h *is unique.*

A consequence of the last corollary is the next result.

Corollary 6. *Let* Ω *be a bounded, convex domain in* $\mathbb{C}^n$ *and let* $a, b \in \Omega$, $a \neq b$. *If* $f \in \mathcal{O}(\mathbb{D}, \Omega)$ *is extremal for* $K_\Omega(a, b)$ *which extends* $\mathcal{C}^k$ *onto* $\mathbb{T}$ *then* f *is also extremal for* $K_\Omega(a', b')$ *and* $\kappa_\Omega(a'; v)$ *for any* $a', b' \in f(\mathbb{D})$ *and* $v \in \mathbb{C}^n$.

Corollary 7. *Let* Ω *be a bounded strongly convex domain with* $\mathcal{C}^2$ *boundary. If* f *is an extremal mapping for the Kobayashi distance, then* $f \in \mathcal{C}^{1/2}(\overline{\mathbb{D}})$.

References

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