PLURISUBHARMONIC FUNCTIONS AS ENVELOPES OF THE RESTRICTIONS OF DISC FUNCTIONALS

Abstract: We consider the set of analytic disc in an open set $X \subset \mathbb{C}^n$ which derivative at the origin is zero and we prove that the envelope of Poisson's disc functional on this set is plurisubharmonic.

INTRODUCTION

Denote by $\mathbb{D}$ the unit disk in $\mathbb{C}$. We recall the definition of the Kobayashi-Royden pseudometric K_X of a domain $X \subset \mathbb{C}^n$.

$$K_X(x,v) = inf\{\alpha > 0, \text{ there is } f \in \mathbb{O}(\mathbb{D}, X), f(0) = x, \alpha f'(0) = v\}$$

where $x \in X$, $v \in \mathbb{C}^n$. And we set $\Omega = \{(x,v) \in X \times \mathbb{C}^n, K_X(x,v) < 1\}$. For $\varphi : X \longrightarrow \mathbb{R}$ upper semicontinuous, $(x,v) \in \Omega$, we define :

$$F_\varphi(x,v) = \inf\left\{\frac{1}{2\pi}\int_0^{2\pi} \varphi o f(e^{i\theta})d\theta, f \in \mathbb{O}(\overline{\mathbb{D}}, X), f(0) = x, f'(0) = v\right\}.$$

Our goal is to check whether F_φ is plurisabharmonic in Ω. For any $x \in X$ we set:

$$EH_\varphi(x) = \inf\left\{\frac{1}{2\pi}\int_0^{2\pi} \varphi o f(e^{i\theta})d\theta, f \in \mathbb{O}(\overline{\mathbb{D}}, X), f(0) = x\right\}.$$

Poletsky has proved that EH_φ is plurisubharmonic and $EH_\varphi = sup\{u \in PSH(X), u \le \varphi\}$. Notice that, $EH_\varphi(x) \le F_\varphi(x,v)$, for any $v \in \mathbb{C}^n$, such that, $(x,v) \in \Omega$.

Theorem 1 *The function $F_\varphi(.,0)$ is plurisubharmonic. Where :*

$$F_\varphi(x,0) = inf\left\{\frac{1}{2\pi}\int_0^{2\pi} \varphi o f(e^{i\theta})d\theta, f \in \mathbb{O}(\overline{\mathbb{D}}, X), f(0) = x, f'(0) = 0\right\}.$$

And for any integer k>0 we have :

$$F_\varphi(x,0) = inf\left\{\frac{1}{2\pi}\int_0^{2\pi} \varphi o f(e^{i\theta})d\theta, f \in \mathbb{O}(\overline{\mathbb{D}}, X), f(0) = x, f^{(1)}(0) = ... = f^{(k)}(0) = 0\right\} = EH_\varphi(x).$$

Before to prove this theorem let's state and prove the following lemma.

Lemma 1 *: Let $\varphi \in C(X)$, then, $F_\varphi \in USC(\Omega)$ and for any, $(x,v) \in \Omega$, we have :*

$$F_\varphi(x,v) = \ inf\ \left\{\frac{1}{2\pi}\int_0^{2\pi} EH_\varphi o f(e^{i\theta})d\theta, f \in \mathbb{O}(\overline{\mathbb{D}}, X), f(0) = x, f'(0) = v\right\}.$$

Proof 1 *:*
$F_\varphi \in USC(\Omega)$.
Let $(x_0, v_0) \in \Omega, c \in \mathbb{R}$ and assume that $F_\varphi(x_0, v_0) < c$. We need to show that there exists an open neighbourhood V, of (x_0, v_0) in Ω such that $F_\varphi(x,v) < c$ for all $(x,v) \in V$.
By definition of enveloppe, there exists $f_0 \in \mathbb{O}(\overline{\mathbb{D}}, X)$, $f_0(0) = x_0$, $f_0'(0) = v_0$, such that:

$$F_\varphi(x_0, v_0) \le \int_{\mathbb{T}} \varphi(f_0(t))d\sigma(t) < c.$$

We choose a compact neighbourhood U *of* (x_0, v_0) *in* Ω, *such that :*
$\{f_0(z) + (x - x_0) + z(v - v_0), z \in \mathbb{D}\} \subset\subset X$ *for all* $(x, v) \in U$ *and define:*

$$f_{x,v}(z) = f_0(z) + x - x_0 + z(v - v_0).$$

Recall that $\varphi \in C(X)$ *and* F *is continuous on* U *where:*

$$F(x, v) = \int_{\mathbb{T}} \varphi(f_0(t) + x - x_0 + t(v - v_0))d\sigma(t).$$

Since $F(x_0, v_0) < c$, *it follows that there exists a neighbourhood* V *of* (x_0, v_0) *contained in* U, *such that for any* $(x, v) \in V$, *we have:* $F_\varphi(x, v) < c$. *Hence* $F_\varphi \in USC(\Omega)$.
We prove the equality.
Let $(x, v) \in \Omega$, *as* $EH_\varphi \leq \varphi$ *then we have :*

$$inf \left\{ \frac{1}{2\pi} \int_0^{2\pi} EH_\varphi o f(e^{i\theta})d\theta, f \in \mathbb{O}(\overline{\mathbb{D}}, X), f(0) = x, f^{'}(0) = v \right\} \leq$$

$$inf \left\{ \frac{1}{2\pi} \int_0^{2\pi} \varphi o f(e^{i\theta})d\theta, f \in \mathbb{O}(\overline{\mathbb{D}}, X), f(0) = x, f^{'}(0) = v \right\} = F_\varphi(x, v).$$

In order to show the other inequality we have to prove that: $F_\varphi(x, v) \leq \frac{1}{2\pi} \int_0^{2\pi} EH_\varphi oh(e^{i\theta})d\theta$ *for any* $h \in \mathbb{O}(\overline{\mathbb{D}}, X)$ *with* $h(0) = x$, $h^{'}(0) = v$. *We will get, this, if we reach to prove that for any* $P \in C(X)$ *with* $EH_\varphi \leq P$ *we have :* $F_\varphi(x, v) \leq \frac{1}{2\pi} \int_0^{2\pi} Poh(e^{i\theta})d\theta$ *for any* $h \in \mathbb{O}(\overline{\mathbb{D}}, X)$ *with* $h(0) = x$, $h^{'}(0) = v$.
Let $\epsilon > 0$, $(x, v) \in \Omega$, $h \in \mathbb{O}(\overline{\mathbb{D}}, X)$ *with* $h(0) = x$, $h^{'}(0) = v$ *and for any* $P \in C(X)$ *with* $EH_\varphi \leq P$. *We construct* $g \in \mathbb{O}(\overline{\mathbb{D}}, X)$, $g(0) = x$, $g^{'}(0) = v$ *such that:*

$$\frac{1}{2\pi} \int_0^{2\pi} \varphi o g(e^{i\theta})d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} Poh(e^{i\theta})d\theta + \epsilon.$$

To simplify the work, for any $f \in \mathbb{O}(\overline{\mathbb{D}}, X)$ *we denote by* $H_\varphi(f)$ *the average of* φ *on the boundary of the analytic disc* f. *In other word* $H_\varphi(f) = \frac{1}{2\pi} \int_0^{2\pi} \varphi oh(e^{i\theta})d\theta$.
Let $(x, v) \in \Omega$, $\epsilon > 0$ *and* $h \in \mathbb{O}(\overline{\mathbb{D}}, X)$ *with* $h(0) = x$, $h^{'}(0) = v$. *For every* $w_0 \in \mathbb{T}$, *Poletsky's theorem ensures that there exists* $r_0 > 1$ *and* $f_0 \in \mathbb{O}(D_{r_0}, X), f_0(0) = h(w_0)$ *such that:*

$$H_\varphi(f_0) \leq P(h(w_0)) + \epsilon.$$

Since $\varphi \in C(X), P \in C(X)$, *there exists an open arc,* $I_0 \ni w_0$ *such that:*

$$f_0(D_{r_0}) + h(w) - h(w_0) \subset\subset X, w \in I_0.$$

and

$$\int_{\mathbb{T}} \varphi(f_0(t) + h(w) - h(w_0))d\sigma(t) \leq P(h(w)) + \epsilon, w \in I_0.$$

We define:

$$F_0 : D_{r_0} \times I_0 \longrightarrow X, \; F_0(z, w) = f_0(z) + h(w) - h(w_0).$$

Then $F_0 \in C^\infty(D_{r_0} \times I_0, X)$, $F_0(., w) \in \mathbb{O}(D_{r_0}, X), F_0(0, w) = h(w)$ *and*

$$H_\varphi(F_0(., w)) \leq P(h(w)) + \epsilon, w \in I_0.$$

A simple compactness argument shows that there exists a cover of $\mathbb{T}$ *by open arcs* $\{I_j\}_{j=1}^N$, $r_j > 1$ *and* $F_j \in C^\infty(D_{r_j} \times I_j, X)$ *with* $F_j(.,w) \in \mathbb{O}(D_{r_j}, X), F_j(0,w) = h(w))$, $F_j(D_{r_j} \times I_j)$ *is relatively compact in X, and*

$$H_\varphi(F_j(.,w)) \leq P(h(w)) + \epsilon, w \in I_j.$$

We set r=*min* r_j, *let K be a compact subset of X containing all the images,* $F_j(D_r \times I_j) \cup h(\overline{\mathbb{D}})$, *and choose*

$$C > max_K(|\varphi| + |P|).$$

There exists disjoint closed arcs $J_j \subset I_j$ *for* $j \in \{1, ..., N\}$ *such that*

$$\sigma(\mathbb{T} \setminus \bigcup_j J_j) < \epsilon/C.$$

Now we choose disjoint open arcs K_j *such that* $J_j \subset K_j \subset I_j$, *and a function* $\rho \in C^\infty(\mathbb{T}, [0,1])$ *satisfying* $\rho(w) = 1$ *if* $w \in \bigcup_{j=1}^N J_j$ *and* $\rho(w) = 0$ *if* $w \in \mathbb{T} \setminus \bigcup_j K_j$, *and finally we define*

$$F(z,w) = \begin{cases} F_j(\rho(w)z, w), & z \in D_r, w \in K_j, j = 1, ..., N \\ h(w), & z \in D_r, w \in \mathbb{T} \setminus \bigcup_j K_j. \end{cases}$$

The choice of ρ *ensures that* $F \in C^\infty(D_r \times \mathbb{T}, X)$ *and* $F(.,w) \in \mathbb{O}(D_r, X), F(0,w) = h(w)$ *for all* $w \in \mathbb{T}$. *Hence*

$$\begin{aligned} &\int_{\mathbb{T}} H_\varphi(F(.,w))d\sigma(w) = \int_{\mathbb{T}\times\mathbb{T}} \varphi(F(z,w))d\sigma(z)d\sigma(w) \\ &\leq \sum_{j=1}^N \int_{J_j} \left(\int_{\mathbb{T}} \varphi(F_j(z,w))d\sigma(z) \right) d\sigma(w) + C\sigma(\mathbb{T} \setminus \bigcup_j J_j) \\ &\leq \sum_{j=1}^N \int_{J_j} H_\varphi(F_j(.,w))d\sigma(w) + \epsilon \leq \sum_{j=1}^N \int_{J_j} P(h(w))d\sigma(w) + \epsilon + \epsilon \\ &\leq \int_{\mathbb{T}} P(h(w))d\sigma(t) + 2\epsilon + C\sigma(\mathbb{T} \setminus \bigcup_j J_j) \leq \int_{\mathbb{T}} P(h(w))d\sigma(w) + 3\epsilon. \end{aligned}$$

Observe that:

$$F(z,w) = h(w) + \sum_{k=-\infty}^{+\infty} \Big(\frac{1}{2\pi} \int_0^{2\pi} (F(z, e^{i\theta}) - h(e^{i\theta}))e^{-ik}d\theta \Big) w^k.$$

Since the map :

$$\mathbb{R} \ni \theta \mapsto F(z, e^{i\theta}) - h(e^{i\theta}) \in \mathbb{C}^n,$$

is infinitely differentiable with period 2π, *the Fourier series converges uniformly on* $\mathbb{T}$ *for any* $z \in D_{r_0}$. *Where* $1 < r_0 < r$.

Now we introduce the j-th partial sums of the series :

$$G_j(z,w) = h(w) + \sum_{k=-j}^{j} \Big(\frac{1}{2\pi} \int_0^{2\pi} (F(z, e^{i\theta}) - h(e^{i\theta}))e^{-ik\theta}d\theta \Big) w^k.$$

Since $G_j \longrightarrow F$ *uniformly on* $D_{r_0} \times \mathbb{T}$ *as* $j \longrightarrow +\infty$ *there exists* j_0 *such that*

$$G_j(D_{r_0} \times \mathbb{T}) \subset X,\ j \geq j_0.$$

and

$$\int_{\mathbb{T}} H_\varphi(G_j(.,w))d\sigma(w) \leq \int_{\mathbb{T}} H_\varphi(F(.,w))d\sigma(w) + \epsilon.$$

Notice that:
$G_j \in \mathbb{O}(D_r \times \mathbb{C}^*, \mathbb{C}^n)$.
$w \mapsto G_j(z,w)$ *has a pole at 0 of order* $\leq j$ *for every* $z \in D_r$.
Since $F(0,e^{i\theta}) = h(e^{i\theta})$ *for all* θ, *the function* $z \mapsto G_j(z,w) - h(w)$ *has a zero at the origin for all* w.
We can write

$$G_j(z,w) = h(w) + zw^{-j}Q_j(z,w),\ \text{where } Q_j \in \mathbb{O}(D_r \times \mathbb{C}, \mathbb{C}^n).$$

For every $k \geq j$ *we get*

$$G_j(zw^k, w) = h(w) + zw^{k-j}Q_j(zw^k, w) \in \mathbb{O}(\mathbb{D} \times \mathbb{D}, \mathbb{C}^n).$$

Since $G_j(z,w) \in X$ *for all* $(z,w) \in \mathbb{D} \times \mathbb{T}$ *and* $j \geq j_0$, *we can choose* $\eta_j \in]0,1[$ *such that*

$$G_j(z,w) \in X,\ (z,w) \in \mathbb{D} \times (\mathbb{D} \setminus D_{\eta_j}).$$

We even have for every $k \geq j$, *that:*

$$G_j(zw^k, w) \in X,\ (z,w) \in \mathbb{D} \times (\mathbb{D} \setminus D_{\eta_j}).$$

Since $G_j(0,w) = h(w) \in X$ *for all* $w \in \overline{\mathbb{D}}$ *there exists* δ_j, *such that*

$$G_j(zw^j, w) \in X,\ (z,w) \in D_{\delta_j} \times \mathbb{D}.$$

If we take $k_j > j$ *such that* $zw^{k_j} \in D_{\delta_j}$ *for* $(z,w) \in \mathbb{D} \times D_{\eta_j}$. *We have*

$$G_j(zw^{k_j}, w) \in X,\ (z,w) \in \mathbb{D} \times D_{\eta_j}.$$

Recall that $j > j_0$ *sufficiently large, we have*

$$G_j(\overline{\mathbb{D}} \times \mathbb{T}) \subset X,$$

$$\int_{\mathbb{T}} H_\varphi(G_j(.,w))d\sigma(w) \leq \int_{\mathbb{T}} H_\varphi(F(.,w))d\sigma(w) + \epsilon.$$

Take $k_j > j+1$ *and set:*

$$G(z,w) = G_j(zw^{k_j}, w) = h(w) + zw^{k_j - j}Q_j(zw^{k_j}, w) \in \mathbb{O}(\overline{\mathbb{D}} \times \overline{\mathbb{D}}, X).$$

We have

$$\int_{\mathbb{T}} H_\varphi(G(.,w))d\sigma(w) = \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} \varphi(G_j(e^{i(\alpha+k_j\theta)}, e^{i\theta}))d\alpha d\theta$$
$$= \frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi} \varphi(G_j(e^{i\alpha}, e^{i\theta}))d\alpha d\theta = \int_{\mathbb{T}} H_\varphi(G_j(.,w))d\sigma(w)$$
$$\leq \int_{\mathbb{T}} H_\varphi(F(.,w))d\sigma(w) + \epsilon.$$

As $\varphi o G$ is continuous on $\mathbb{T}\times\mathbb{T}$. We set

$$\varphi o G(z_0,w_0)=\varphi o G_j(z_0w_0^{k_j},w_0)=min\{\varphi o G_j(zw^{k_j},w),(z,w)\in\mathbb{T}\times\mathbb{T}\},$$

and we take

$$g(w)=G_j(w^{k_j}e^{i\alpha_0},w).$$

Notice that $g\in\mathbb{O}(\overline{\mathbb{D}},X), g(0)=h(0), g^{'}(0)=h^{'}(0)$ and

$$H_\varphi(g)=\int_0^{2\pi}\varphi(G_j(e^{i\alpha_0}e^{ik_j\theta},e^{i\theta}))d\theta\le\frac{1}{2\pi}\int_0^{2\pi}\int_0^{2\pi}\varphi(G_j(e^{i\alpha}e^{ik_j\theta},e^{i\theta}))d\alpha d\theta$$

$$=\int_{\mathbb{T}}H_\varphi(G_j(.,w))d\sigma(w)\le\int_{\mathbb{T}}H_\varphi(F(.,w))d\sigma(w)+\epsilon\le\int_{\mathbb{T}}P(h(w))d\sigma(w)+4\epsilon.$$

So $F_\varphi(x,v)\le\int_{\mathbb{T}}P(h(w))d\sigma(w)+4\epsilon$ for any $\epsilon>0$ hence $F_\varphi(x,v)\le\int_{\mathbb{T}}P(h(w))d\sigma(w)$. We have this for any continuous function P bigger than EH_φ by monotonie convergence theorem we get $F_\varphi(x,v)\le\frac{1}{2\pi}\int_0^{2\pi}EH_\varphi oh(e^{i\theta})d\theta$. As h is arbitrary so we have:

$$F_\varphi(x,v)\le\ inf\left\{\frac{1}{2\pi}\int_0^{2\pi}EH_\varphi of(e^{i\theta})d\theta, f\in\mathbb{O}(\overline{\mathbb{D}},X), f(0)=x, f^{'}(0)=v)\right\}.$$

Proof of theorem1.

Proof 2 *For $x\in X$ if we consider the constant disc $f=x$, Lemma 1, gives, $F_\varphi(x,0)\le EH_\varphi(x)$, hence $F_\varphi(x,0)=EH_\varphi(x)$. So $F_\varphi(.,0)\in PSH(X)$ and it coincides with the largest plurisubhamonic function less than φ.*
Let $k>0$, $x\in X$, $h\in\mathbb{O}(\overline{\mathbb{D}},X)$ with $h(0)=x$, $h^{(1)}(0)=...=h^{(k)}(0)=0$ if we repeat the proof of Lemma 1, we get,

$$inf\left\{\frac{1}{2\pi}\int_0^{2\pi}\varphi of(e^{i\theta})d\theta, f\in\mathbb{O}(\overline{\mathbb{D}},X), f(0)=x, f^{(1)}(0)=...=f^{(k)}(0)=0\right\}\le\frac{1}{2\pi}\int_0^{2\pi}EH_\varphi oh(e^{i\theta})d\theta.$$

On considering the constant disc we get

$$inf\left\{\frac{1}{2\pi}\int_0^{2\pi}\varphi of(e^{i\theta})d\theta, f\in\mathbb{O}(\overline{\mathbb{D}},X), f(0)=x, f^{(1)}(0)=...=f^{(k)}(0)=0\right\}=EH_\varphi(x)=F_\varphi(x,0).$$