A COUNTEREXAMPLE TO BREMERMANN'S THEOREM ON SHILOV BOUNDARIES

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**Definition 1.** Let $D \subset \mathbb{C}^n$ be a bounded domain. Define

$$\mathcal{A}(D) := \mathcal{C}(\overline{D}) \cap \mathcal{O}(D), \quad \mathcal{B}(D) := \text{cl}_{\mathcal{C}(\overline{D})} \mathcal{O}(\overline{D}).$$

Let $\partial_S D$ (resp. $\partial_B D$) be the Shilov (resp. Bergman) boundary of $D$, i.e. the minimal compact set $K \subset \overline{D}$ such that $\max_{\overline{D}} |f| = \max_K |f|$ for every $f \in \mathcal{A}(D)$ (resp. $f \in \mathcal{B}(D)$).

One can easily prove that in the case of the Bergman boundary it suffices to check that $\max_K |f| = \max_{\overline{D}} |f|$ for every $f \in \mathcal{O}(\overline{D})$.

Observe that $\mathcal{B}(D) \subset \mathcal{A}(D)$. In particular, $\partial_B D \subset \partial_S D \subset \partial D$. Notice that in general $\partial_B D \not\subset \partial_S D$, e.g. if $D := \{(z, w) \in \mathbb{D} \times \mathbb{C} : |w| < |z|^{-\log |z|}\} \subset \mathbb{D} \times \mathbb{C}$, then $\mathbb{T}^2 = \overline{\mathbb{D}} \times \mathbb{C} : |w| = |z|^{-\log |z|} \in \partial_S D$.

Assume that the envelope of holomorphy $\tilde{D}$ of $D$ is univalent. It is well-known that $\sup_{\overline{D}} |g| = \sup_D |g|$ for every $g \in \mathcal{O}(\overline{D})$. In particular, $\tilde{D}$ is bounded.

Take a $g \in \mathcal{A}(\tilde{D})$ (resp. $g \in \mathcal{B}(\tilde{D})$) and let $C := \max_{\overline{D}} |g|$ (resp. $C := \max_{\overline{D}} |g|$). Since $\mathcal{A}(\tilde{D}) |_{\overline{D}} \subset \mathcal{A}(D)$ (resp. $\mathcal{O}(\overline{D}) |_{\overline{D}} \subset \mathcal{O}(D)$), we get $|g| \leq C$ on $D$. Hence $|g| \leq C$ on $\tilde{D}$. Consequently, $\partial_S \tilde{D} \subset \partial_S D$ (resp. $\partial_B \tilde{D} \subset \partial_B D$).

In [Bre 1959], p. 259, H. J. Bremermann wrote: On the other hand it is trivial that $\partial_S D \subset \partial_S \tilde{D}$.

**Remark 2.**

(a) The inclusion $\partial_S D \subset \partial_S \tilde{D}$ is really trivial if we additionally assume that $\mathcal{A}(D) \subset \mathcal{A}(\tilde{D}) |_{\overline{D}}$.

(b) There exists a Reinhardt domain $D \subset \mathbb{C}_+ \times \mathbb{C}$ such that $\mathcal{A}(D) \not\subset \mathcal{A}(\tilde{D}) |_{\overline{D}}$ (cf. [Kos-Zwo 2013], Example 6.1).

(c) $\partial_S D \subset \partial_S \tilde{D}$ for every bounded Reinhardt domain ([Kos-Zwo 2013], Corollary 6.2).

(d) If we additionally assume that $\mathcal{O}(\overline{D}) \subset \mathcal{O}(\overline{\tilde{D}}) |_{\overline{D}}$, then $\partial_B D \subset \partial_B \tilde{D}$.

(e) If we additionally assume that $\overline{D}$ has a neighborhood basis consisting of domains with univalent envelopes of holomorphy, then $\mathcal{O}(\overline{D}) \subset \mathcal{O}(\overline{\tilde{D}}) |_{\overline{D}}$. Consequently, $\partial_B D = \overline{\partial_B \tilde{D}}$.

Indeed, take an $f \in \mathcal{O}(\overline{D})$, say $f \in \mathcal{O}(G)$, where $G$ is a bounded domain with univalent envelope of holomorphy $\tilde{G}$ with $\overline{D} \subset G$. Then $f$ extends to an $\tilde{f} \in \mathcal{O}(\overline{\tilde{D}})$. We only need to show that $\overline{D} \subset \tilde{G}$.

It is known (cf. e.g. [Jar-Pfl 2000], Theorem 1.10.4(iii)) $\text{dist}(\overline{\mathcal{D}}_{\mathcal{O}(\tilde{g})}, \partial \tilde{G}) = \text{dist}(\overline{D}, \partial \tilde{G})$. It remains to observe that $\tilde{D} \subset \overline{D}_{\mathcal{O}(\tilde{g})}$.

(f) If $D \subset \mathbb{C}^n$ is bounded balanced domain, then $\partial_B D = \partial_B \tilde{D}$.

Indeed, first recall (cf. e.g. [Jar-Pfl 2000], Remark 1.9.6(e)) that the envelope of holomorphy of a balanced domain is always a balanced domain, in particular, it is univalent. Moreover, $(\overline{D} + \mathbb{B}(\varepsilon))_{\varepsilon > 0}$ forms a neighborhood basis of $\overline{D}$ consisting of bounded balanced domains.

(g) It is not known whether $\partial_S D = \partial_S \tilde{D}$ for bounded balanced domains $D$.

**Theorem 3.** There exists a bounded Hartogs domain $D \subset \mathbb{C}^2$ with a univalent envelope of holomorphy $\tilde{D}$ such that:

- $\partial_S D \neq \partial_S \tilde{D}$,
\[ \partial_B D \neq \partial_B \overline{D}, \]
\[ \mathcal{O}(\overline{D}) \subset \mathcal{A}(\overline{D})_{\overline{T}}. \]

**Proof.** Let \( A := \{ z \in \mathbb{C} : 1/2 < |z| < 1 \}, I := [1/2, 1] \). Then we introduce
\[
D := \{ (re^{i\theta}, w) \in (A \setminus I) \times \mathbb{D}(3) : \\
\varphi \in (0, \pi/2) \implies |w| < 1, \\
\varphi \in (\pi/2, 3\pi/2) \implies |w| < 3, \\
\varphi \in [3\pi/2, 2\pi) \implies 2 < |w| < 3 \} =: D_1 \cup D_{2,3} \cup D_4.
\]

Note that \( D \) is a Hartogs domain over \( A \setminus I \) with connected circular fibers and \( D \cap ((A \setminus I) \times \{ 0 \}) \neq \emptyset \). Then Corollary 3.1.10(b) in [Jar-Pfl 2000] implies that \( D \) has a univalent envelope of holomorphy \( D \). Moreover, using the Cauchy integral formula shows that \( D \) contains the domain
\[ \{(re^{i\theta}, w) \in (A \setminus I) \times \mathbb{D}(3) : \varphi \in (\pi/2, 2\pi) \}. \]

In particular, \( \{x\} \times \mathbb{D}(3) \subset \overline{D} \), \( x \in I \). Therefore, if \( f \in \mathcal{A}(\overline{D}) \), then \( f(x, \cdot) \in \mathcal{A}(\mathbb{D}(3)) \), \( x \in I \) (use the Weierstrass theorem). Hence, by maximum principle, \( (I \times \mathbb{D}(3)) \cap \partial_S \overline{D} = \emptyset \).

Fix \( 0 < \varepsilon \ll 1/2 \) and put \( A' := \{ z \in \mathbb{C} : 1/2 - \varepsilon < |z| < 1 + \varepsilon \}, \)
\[
D' := \{ (re^{i\theta}, w) \in A \times \mathbb{D}(3 + \varepsilon) : \\
\varphi \in (-\varepsilon, \pi/2) \implies |w| < 1 + \varepsilon, \\
\varphi \in (\pi/2 - \varepsilon, 3\pi/2 + \varepsilon) \implies |w| < 3 + \varepsilon, \\
\varphi \in [3\pi/2, 2\pi + \varepsilon) \implies 2 - \varepsilon < |w| < 3 + \varepsilon \} = D'_1 \cup D'_{2,3} \cup D'_4.
\]

Observe that \( \overline{D} \subset D' \).

Let \( L_1 \) (resp. \( L_2 \)) denote the branch of the logarithm defined on \( \mathbb{C} \setminus \{ x + iy \in \mathbb{C} : x \geq 0, y = x \} \) (resp. \( \mathbb{C} \setminus \{ x + iy \in \mathbb{C} : x \geq 0, y = -x \} \)) such that \( L_1(-1/2) = L_2(-1/2) = \log(1/2) + \pi i \). Now we define \( g : D' \to \mathbb{C} \):
\[
g(z, w) := \begin{cases} L_1(z), & \text{if } (z, w) \in D', |w| > 1.4 \\
L_2(z), & \text{if } (z, w) \in D', |w| \leq 1.6 \end{cases}
\]

Observe that \( g \) is well defined and \( g \in \mathcal{O}(D') \). Put \( f := g|_{\overline{D}} \). Then \( f \in \mathcal{O}(\overline{D}) \) and \( f(z, w) = \log(-z) + \pi i \), \( (z, w) \in D \). Thus
\[
f(x, w) = \begin{cases} \log x, & \text{if } |w| \leq 1 \\
\log x + 2\pi i, & \text{if } 2 \leq |w| \leq 3 \end{cases}, \quad x \in I.
\]

Finally, we observe that \( h := \exp(if + 2\pi) \in \mathcal{O}(\overline{D}), \)
\[
|h(x, w)| = \begin{cases} e^{2\pi}, & \text{if } |w| \leq 1 \\
1, & \text{if } 2 \leq |w| \leq 3 \end{cases}, \quad x \in I,
\]
and \( |h| = e^{2\pi x + 2\pi} \leq e^{2\pi} \) on the remaining part of \( \overline{D} \). Therefore, \( (I \times \mathbb{D}) \cap \partial_B D \neq \emptyset \). In particular, \( f \notin \mathcal{A}(\overline{D})_{\overline{T}} \) and \( \partial_B \overline{D} \subset \partial_S \overline{D} \subset \partial_B D \subset \partial_S D \).

**References**

