A COUNTEREXAMPLE TO BREMERMANN'S THEOREM ON SHILOV BOUNDARIES

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Definition 1. Let $D \subset \mathbb{C}^n$ be a bounded domain. Define

$$\mathcal{A}(D) := \mathcal{C}(\overline{D}) \cap \mathcal{O}(D), \quad \mathcal{B}(D) := \operatorname{cl}_{\mathcal{C}(\overline{D})} \mathcal{O}(\overline{D}).$$

Let $\partial_S D$ (resp. $\partial_B D$) be the *Shilov* (resp. *Bergman*) *boundary* of D, i.e. the minimal compact set $K \subset \overline{D}$ such that $\max_K |f| = \max_{\overline{D}} |f|$ for every $f \in \mathcal{A}(D)$ (resp. $f \in \mathcal{B}(D)$).

One can easily prove that in the case of the Bergman boundary it suffices to check that $\max_K |f| = \max_{\overline{D}} |f|$ for every $f \in \mathcal{O}(\overline{D})$.

Obviously, $\mathcal{B}(D) \subset \mathcal{A}(D)$. In particular, $\partial_B D \subset \partial_S D \subset \partial D$. Notice that in general $\partial_B D \subsetneq \partial_S D$, e.g. if $D := \{(z, w) \in \mathbb{D}_* \times \mathbb{C} : |w| < |z|^{-\log|z|}\} \subset \mathbb{D}_* \times \mathbb{D}$, then $\mathbb{T}^2 = \partial_B D \subsetneq \{\overline{\mathbb{D}} \times \mathbb{C} : |w| = |z|^{-\log|z|}\} = \partial_S D$.

Assume that the envelope of holomorphy $\widetilde{D}$ of D is univalent. It is well-known that $\sup_{\widetilde{D}} |g| = \sup_D |g|$ for every $g \in \mathcal{O}(\widetilde{D})$. In particular, $\widetilde{D}$ is bounded.

Take a $g \in \mathcal{A}(\widetilde{D})$ (resp. $g \in \mathcal{B}(\widetilde{D})$) and let $C := \max_{\partial_S D} |g|$ (resp. $C := \max_{\partial_B D} |g|$). Since $\mathcal{A}(\widetilde{D})|_{\overline{D}} \subset \mathcal{A}(D)$ (resp. $\mathcal{O}(\overline{\widetilde{D}})|_{\overline{D}} \subset \mathcal{O}(\overline{D})$), we get $|g| \le C$ on D. Hence $|g| \le C$ on $\widetilde{D}$. Consequently, $\partial_S \widetilde{D} \subset \partial_S D$ (resp. $\partial_B \widetilde{D} \subset \partial_B D$).

In [Bre 1959], p. 259, H. J. Bremermann wrote: *On the other hand it is trivial that* $\partial_S D \subset \partial_S \widetilde{D}$.

Remark 2. (a) The inclusion $\partial_S D \subset \partial_S \widetilde{D}$ is really trivial if we additionally assume that $\mathcal{A}(D) \subset \mathcal{A}(\widetilde{D})|_{\overline{D}}$.

(b) There exists a Reinhardt domain $D \subset \mathbb{C}_* \times \mathbb{C}$ such that $\mathcal{A}(D) \not\subset \mathcal{A}(\widetilde{D})|_{\overline{D}}$ (cf. [Kos-Zwo 2013], Example 6.1).

(c) $\partial_S D \subset \partial_S \widetilde{D}$ for every bounded Reinhardt domain ([Kos-Zwo 2013], Corollary 6.2).

(d) If we additionally assume that $\mathcal{O}(\overline{D}) \subset \mathcal{O}(\overline{\widetilde{D}})|_{\overline{D}}$, then $\partial_B D \subset \partial_B \widetilde{D}$.

(e) If we additionally assume that $\overline{D}$ has a neighborhood basis consisting of domains with univalent envelopes of holomorphy, then $\mathcal{O}(\overline{D}) \subset \mathcal{O}(\overline{\widetilde{D}})|_{\overline{D}}$. Consequently, $\partial_B D = \partial_B \widetilde{D}$.

Indeed, take an $f \in \mathcal{O}(\overline{D})$, say $f \in \mathcal{O}(G)$, where G is a bounded domain with univalent envelope of holomorphy $\widetilde{G}$ with $\overline{D} \subset G$. Then f extends to an $\widetilde{f} \in \mathcal{O}(\widetilde{G})$. We only need to show that $\overline{\widetilde{D}} \subset \widetilde{G}$.

It is known (cf. e.g. [Jar-Pfl 2000], Theorem 1.10.4(iii)) $\operatorname{dist}(\widehat{\overline{D}}_{\mathcal{O}(\widetilde{G})}, \partial\widetilde{G}) = \operatorname{dist}(\overline{D}, \partial\widetilde{G})$. It remains to observe that $\widetilde{D} \subset \widehat{\overline{D}}_{\mathcal{O}(\widetilde{G})}$.

(f) If $D \subset \mathbb{C}^n$ is bounded balanced domain, then $\partial_B D = \partial_B \widetilde{D}$.

Indeed, first recall (cf. e.g. [Jar-Pfl 2000], Remark 1.9.6(e)) that the envelope of holomorphy of a balanced domain is always a balanced domain, in particular, it is univalent. Moreover, $(\overline{D} + \mathbb{B}(\varepsilon))_{\varepsilon>0}$ forms a neighborhood basis of $\overline{D}$ consisting of bounded balanced domains.

(g) [?] It is not known whether $\partial_S D = \partial_S \widetilde{D}$ for bounded balanced domains D [?]

Theorem 3. *There exists a bounded Hartogs domain $D \subset \mathbb{C}^2$ with a univalent envelope of holomorphy $\widetilde{D}$ such that:*

- $\partial_S D \neq \partial_S \widetilde{D}$,

- $\partial_B D \neq \partial_B \widetilde{D}$,
- $\mathcal{O}(\overline{D}) \not\subset \mathcal{A}(\widetilde{D})|_{\overline{D}}$.

Proof. Let $A := \{z \in \mathbb{C} : 1/2 < |z| < 1\}$, $I := [1/2, 1]$. Then we introduce

$$\begin{aligned} D :&= \{(re^{i\varphi}, w) \in (A \setminus I) \times \mathbb{D}(3) : \\ &\varphi \in (0, \pi/2] \Longrightarrow |w| < 1, \\ &\varphi \in (\pi/2, 3\pi/2) \Longrightarrow |w| < 3, \\ &\varphi \in [3\pi/2, 2\pi) \Longrightarrow 2 < |w| < 3\} =: D_1 \cup D_{2,3} \cup D_4. \end{aligned}$$

Note that D is a Hartogs domain over $A \setminus I$ with connected circular fibers and $D \cap ((A \setminus I) \times \{0\}) \neq \varnothing$. Then Corollary 3.1.10(b) in [Jar-Pfl 2000] implies that D has a univalent envelope of holomorphy $\widetilde{D}$. Moreover, using the Cauchy integral formula shows that $\widetilde{D}$ contains the domain

$$\{(re^{i\varphi}, w) \in (A \setminus I) \times \mathbb{D}(3) : \varphi \in (\pi/2, 2\pi)\}.$$

In particular, $\{x\} \times \mathbb{D}(3) \subset \overline{\widetilde{D}}$, $x \in I$. Therefore, if $f \in \mathcal{A}(\widetilde{D})$, then $f(x, \cdot) \in \mathcal{A}(\mathbb{D}(3))$, $x \in I$ (use the Weierstrass theorem). Hence, by maximum principle, $(I \times \mathbb{D}(3)) \cap \partial_S \widetilde{D} = \varnothing$.

Fix $0 < \varepsilon \ll 1/2$ and put $A' := \{z \in \mathbb{C} : 1/2 - \varepsilon < |z| < 1 + \varepsilon\}$,

$$\begin{aligned} D' :&= \{(re^{i\varphi}, w) \in A \times \mathbb{D}(3 + \varepsilon) : \\ &\varphi \in (-\varepsilon, \pi/2] \Longrightarrow |w| < 1 + \varepsilon, \\ &\varphi \in (\pi/2 - \varepsilon, 3\pi/2 + \varepsilon) \Longrightarrow |w| < 3 + \varepsilon, \\ &\varphi \in [3\pi/2, 2\pi + \varepsilon) \Longrightarrow 2 - \varepsilon < |w| < 3 + \varepsilon\} = D'_1 \cup D'_{2,3} \cup D'_4. \end{aligned}$$

Observe that $\overline{D} \subset D'$.

Let L_1 (resp. L_2) denote the branch of the logarithm defined on $\mathbb{C} \setminus \{x + iy \in \mathbb{C} : x \geq 0,\ y = x\}$ (resp. $\mathbb{C} \setminus \{x + iy \in \mathbb{C} : x \geq 0,\ y = -x\}$) such that $L_1(-1/2) = L_2(-1/2) = \log(1/2) + \pi i$. Now we define $g : D' \longrightarrow \mathbb{C}$:

$$g(z, w) := \begin{cases} L_1(z), & \text{if } (z, w) \in D',\ |w| > 1.4 \\ L_2(z), & \text{if } (z, w) \in D',\ |w| < 1.6 \end{cases}.$$

Observe that g is well defined and $g \in \mathcal{O}(D')$. Put $f := g|_{\overline{D}}$. Then $f \in \mathcal{O}(\overline{D})$ and $f(z, w) = \operatorname{Log}(-z) + \pi i$, $(z, w) \in D$. Thus

$$f(x, w) = \begin{cases} \log x, & \text{if } |w| \leq 1 \\ \log x + 2\pi i, & \text{if } 2 \leq |w| \leq 3 \end{cases}, \quad x \in I.$$

Finally, we observe that $h := \exp(if + 2\pi) \in \mathcal{O}(\overline{D})$,

$$|h(x, w)| = \begin{cases} e^{2\pi}, & \text{if } |w| \leq 1 \\ 1, & \text{if } 2 \leq |w| \leq 3 \end{cases}, \quad x \in I,$$

and $|h| = e^{\operatorname{Im} f + 2\pi} < e^{2\pi}$ on the remaining part of $\overline{D}$. Therefore, $(I \times \overline{\mathbb{D}}) \cap \partial_B D \neq \varnothing$. In particular, $f \notin \mathcal{A}(\widetilde{D})|_{\overline{D}}$ and $\partial_B \widetilde{D} \subset \partial_S \widetilde{D} \subsetneq \partial_B D \subset \partial_S D$. □

References

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