NON-NATURAL FRÉCHET SPACES OF HOLOMORPHIC FUNCTIONS

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Based [Vog 2013].

We assume that all the complex manifolds which will appear below are countable at infinity.

Definition 1. For a complex manifold X, let τ_X denote the topology of almost uniform convergence on $\mathcal{O}(X)$, i.e. the topology generated by the seminorms

$$\mathcal{O}(X) \ni f \longmapsto \|f\|_K := \sup_K |f|, \quad K \subset\subset X.$$

It is well know that $(\mathcal{O}(X), \tau_X)$ is a Fréchet space. Suppose that $(\mathcal{F}, \tau)$ is another *Fréchet space in* $\mathcal{O}(X)$, i.e. $\mathcal{F} \subset \mathcal{O}(X)$ and $(\mathcal{F}, \tau)$ is a Fréchet space. We say that $(\mathcal{F}, \tau)$ is a *natural Fréchet space in* $\mathcal{O}(X)$ if the identity mapping $\mathrm{id} : (\mathcal{F}, \tau) \longrightarrow (\mathcal{O}(X), \tau_X)$ is continuous.

Remark 2. Let $(\mathcal{F}, \tau)$ be a Fréchet space in $\mathcal{O}(X)$. Then $(\mathcal{F}, \tau)$ is natural iff for every sequence $(f_s)_{s=0}^\infty \subset \mathcal{F}$ we have: $f_s \overset{\tau}{\longrightarrow} f_0 \Longrightarrow f_s \longrightarrow f_0$ pointwise on X.

The question whether there are non-natural Fréchet spaces $(\mathcal{F}, \tau)$ in $\mathcal{O}(X)$ remained open for many years (cf. [Jar-Pfl 2008], Remark 1.10.6(b)).

Proposition 3. *If X is a connected complex manifold and $(\mathcal{F}, \tau)$ is a natural Fréchet space in $\mathcal{O}(X)$, then there exists a continuous norm on $\mathcal{F}$.*

Proof. Let $(q_i)_{i=1}^\infty$ be a family of seminorms on $\mathcal{F}$ that generates τ. Take a compact $K \subset X$ with $\mathrm{int}\, K \neq \varnothing$. The continuity of $\mathrm{id} : (\mathcal{F}, \tau) \longrightarrow (\mathcal{O}(X), \tau_X)$ implies that there exist $C > 0$, $n \in \mathbb{N}$, and $i_1, \dots, i_n \in \mathbb{N}$ such that

$$\|f\|_K \le C \max\{q_{i_1}(f), \dots, q_{i_n}(f)\}, \quad f \in \mathcal{F}.$$

Consequently, by the identity principle for holomorphic functions, $\| \ \|_K$ is a continuous norm on $\mathcal{F}$. □

Definition 4. Let $\omega := \mathbb{C}^{\mathbb{N}}$. Put

$$p_n(x) := \max\{|x_1|, \dots, |x_n|\}, \quad x = (x_i)_{i=1}^\infty \in \omega, \quad n \in \mathbb{N}.$$

It is well know that ω endowed with the topology generated by the seminorms $(p_n)_{n=1}^\infty$ is a Fréchet space.

Proposition 5. *There are no continuous norms on ω.*

Proof. Suppose that q is such a norm. Then there exist $C > 0$ and $n \in \mathbb{N}$ such that $q \le C p_n$. In particular, $q(x) = 0$ if $x_1 = \dots = x_n = 0$ — a contradiction. □

Theorem 6. *Let X be a connected complex manifold and let $\mathcal{F} \subset \mathcal{O}(X)$ be a vector space such that* $\dim \mathcal{F} \ge \dim \omega$*. Then on $\mathcal{F}$ there exists a Fréchet topology τ such that $(\mathcal{F}, \tau)$ is not natural.*

Proof. Let $(a_s)_{s=1}^\infty \subset X$ be dense. Define a linear mapping $I : \mathcal{F} \longrightarrow \omega$, $I(f) := (f(a_s))_{s=1}^\infty$. Since the sequence is dense, the operator I is injective. In particular, $\dim \mathcal{F} \le \dim \omega$. Consequently, there exists a linear isomorphism $L : \mathcal{F} \longrightarrow \omega$. Now, using L, we transport the topology of ω to $\mathcal{F}$. It remains to apply Propositions 3 and 5. □

Thus, the main question is when $\dim \mathcal{F} \ge \dim \omega$. Using Functional Analysis methods we get the following general answer.

Theorem 7 (Main result)**.** *Let X be a connected complex manifold and let $(\mathcal{F},\tau')$ be a Fréchet space such that $\mathcal{F}\subset\mathcal{O}(X)$ and $(\mathcal{F},\tau')$ is not Banach (e.g. $(\mathcal{F},\tau')=(\mathcal{O}(X),\tau_X)$). Then on $\mathcal{F}$ there exists a Fréchet topology τ such that $(\mathcal{F},\tau)$ is not natural.*

Proof. By Eidelheit's theorem (cf. [Mei-Vog 1997], § 26.28) we get a linear surjective mapping $S:\mathcal{F}\longrightarrow\omega$. Hence $\dim\mathcal{F}\geq\dim\omega$ and we apply Theorem 6. □

[?] It would be nice to prove Theorem 7 using only complex analysis methods [?]
Using only complex analysis methods we get the following weaker result.

Theorem 8. *If X is a connected complex manifold with $\mathcal{O}(X)\neq\mathbb{C}$, then there exists a Fréchet topology τ on $\mathcal{O}(X)$ such that $(\mathcal{O}(X),\tau)$ is not natural.*

Proof. Fix an arbitrary non-constant function $f_0\in\mathcal{O}(X)$. Then $D:=f(X)$ is a planar domain. Let $(z_s)_{s=1}^{\infty}\subset D$ be a sequence without accumulations points in D. For every $s\in\mathbb{N}$ fix a point $b_s\in f_0^{-1}(z_s)$. Define a linear mapping $S:\mathcal{O}(X)\longrightarrow\omega$, $S(f):=(f(b_s))_{s=1}^{\infty}$. Then S is surjective. In fact, it is well known that for any sequence $x=(x_s)_{s=1}^{\infty}\in\omega$ there exists a function $\varphi\in\mathcal{O}(D)$ such that $f(z_s)=x_s$, $s\in\mathbb{N}$. Put $f:=\varphi\circ f_0$. Then $S(f)=x$.

Thus $\dim\omega\leq\dim\mathcal{O}(X)$ and we apply Theorem 6. □

Remark 9. Let X be a disconnected complex manifold and let $X=X_1\cup X_2$, where X_1, X_2 are non-empty open and disjoint. Let $(\mathcal{F}_i,\tau_i)$ be a Fréchet space in $\mathcal{O}(X_i)$, $i=1,2$. Define

$$\mathcal{F}:=\{f\in\mathcal{O}(X):f|_{X_i}\in\mathcal{F}_i,\ i=1,2\}.$$

Observe that in fact $\mathcal{F}\simeq\mathcal{F}_1\times\mathcal{F}_2$. We define a topology τ on $\mathcal{F}$. If $(q_{i,k})_{k=1}^{\infty}$ is a family of seminorms generating τ_i, $i=1,2$, then τ is generated by the seminorms

$$\mathcal{F}\ni f\longmapsto q_{1,k}(f|_{X_1})+q_{2,\ell}(f|_{X_2}),\quad k,\ell\in\mathbb{N}.$$

Equivalently, for $(f_s)_{s=0}^{\infty}\subset\mathcal{F}$ we have

$$f_s\overset{\tau}{\longrightarrow}f_0:\Longleftrightarrow f_s|_{X_i}\overset{\tau_i}{\longrightarrow}f_0|_{X_i},\ i=1,2.$$

Then:

- $(\mathcal{F},\tau)$ is a Fréchet space in $\mathcal{O}(X)$,
- $(\mathcal{F},\tau)$ is natural in $\mathcal{O}(X)$ iff $(\mathcal{F}_i,\tau_i)$ is natural in $\mathcal{O}(X_i)$, $i=1,2$,
- $(\mathcal{F},\tau)$ is Banach in $\mathcal{O}(X)$ iff $(\mathcal{F}_i,\tau_i)$ is Banach in $\mathcal{O}(X_i)$, $i=1,2$.

Consequently, if X_1 is connected and $(\mathcal{F},\tau_1)$ is a non-natural Fréchet space in $\mathcal{O}(X_1)$ (produced, for instance, via Theorem 7), then for arbitrary Fréchet space $(\mathcal{F}_2,\tau_2)$ in $\mathcal{O}(X_2)$ we get a non-natural Fréchet space $(\mathcal{F},\tau)$ in $\mathcal{O}(X)$.

References

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