EXTENSION OF CR-FUNCTIONS ON GENERALIZED TUBE DOMAINS (BASED ON PAPER BY A. BOGGESS, R. DWILEWICZ AND Z. SŁODKOWSKI)

TOMASZ WARSZAWSKI

Let $M \subset \mathbb{C}^n$ be a real submanifold and let $f : M \longrightarrow \mathbb{C}$ be $\mathcal{C}^1$. Then f is said to be a CR-function if

$$\sum_{j=1}^{n} a_j \frac{\partial f}{\partial \overline{z}_j}(z) = 0$$

for any $a \in T_M^{\mathbb{C}}(z)$ and $z \in M$.

The following results from [2] are presented.

Theorem 1. *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a $\mathcal{C}^\infty$ domain with connected boundary. Moreover, assume that*

$$\Omega \subset \{z \in \mathbb{C}^n : |\operatorname{Re} z| \leq |\operatorname{Im} z| - \Phi(\operatorname{Im} z) + C\},$$

where $C > 0$ and $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}_{>0}$ has bounded sublevels $\{\Phi \leq c\}$.

Then any $\mathcal{C}^\infty$-smooth CR-function on $\partial\Omega$ extends to a function from class $\mathcal{O}(\Omega) \cap \mathcal{C}^\infty(\mathbb{C}^n)$.

Theorem 2. *Let $\Omega \subset \mathbb{C}^n$, $n \geq 2$, be a $\mathcal{C}^\infty$ domain with connected boundary. Moreover, assume that there is a function $F \in \mathcal{O}(\mathbb{C}^n)$ and a neighborhood U of $\mathbb{C}^n \setminus \Omega$ such that*

(1) $\operatorname{Re} F < 0$ *on* Ω
(2) $F' \neq 0$ *on* U
(3) *putting* $A_z := \{\zeta \in \mathbb{C}^n : F(\zeta) = F(z)\}$ *we have that* $A_z \cap \overline{\Omega}$ *is compact for* $z \in \overline{\Omega} \cap U$.

Then any $\mathcal{C}^\infty$-smooth CR-function on $\partial\Omega$ extends to a function from class $\mathcal{O}(\Omega) \cap \mathcal{C}^\infty(\mathbb{C}^n)$.

We omit in this note the proof of Theorem 2 $\Longrightarrow$ Theorem 1, which is elementary and will be shown in the talk.

Sketch of proof of Theorem 2. Let f be a given function and let an open set $U' \subset U$ with $\partial U' \cap \partial U = \emptyset$ satisfy the same as U. By [1, Lemma 2.2] there exists $\widetilde{f} \in \mathcal{C}^\infty(\mathbb{C}^n)$ such that $\widetilde{f}|_{\partial\Omega} = f$, $\operatorname{supp} \widetilde{f} \subset U' \cap \overline{\Omega}$ and $\omega := \overline{\partial}\widetilde{f}$ vanishes to infinity order on $\partial\Omega$. If there is $u \in \mathcal{C}^\infty(\mathbb{C}^n)$ such that $\overline{\partial}u = \omega$ on $\mathbb{C}^n$ and $u = 0$ outside Ω then $f_1 := \widetilde{f} - u$ is the desired extension.

Let $g \in \mathcal{O}(\mathbb{C}^{2n}, \mathbb{C}^n)$ be defined by

$$F(\zeta) - F(z) = g(\zeta, z) \bullet (\zeta - z)$$

and let

$$A := \{(\zeta, z) \in U \times \mathbb{C}^n : g(\zeta, z) \bullet (\zeta - z) = 0\}.$$

Now we use the theory described in [3]. For $X, Y \subset \mathbb{C}^N$ being a real $\mathcal{C}^\infty$ oriented manifolds of dimensions n, m we define:

$\mathcal{D}^p(X)$:= set of $\mathcal{C}^\infty$ forms on X of degree p with compact support; $\mathcal{D}(X) := \bigcup_p \mathcal{D}^p(X)$.

${\mathcal{D}'}^r(X)$:= set of currents on X of degree r = set of forms on X of degree r with distribution coefficients = the dual of $\mathcal{D}^{n-r}(X)$; $\mathcal{D}'(X) := \bigcup_r {\mathcal{D}'}^r(X)$.

Kernel := any element of ${\mathcal{D}'}^{m+r}(Y \times X)$.

With a kernel K we associate the following operator on forms on Y, denoted again by K

$$K(\varphi)(x) = \int_Y K(y, x) \wedge \varphi(y)$$

if RHS is well-defined (e.g. when φ has compact support and K is continuous).

By Theorem 5.8 there exists a $\mathcal{C}^\infty$ kernel $S \in \mathcal{D}'(U \times \mathbb{C}^n)$ of complex type $(n, n-1)$ such that $\overline{\partial} S$ is the identity as the operator on forms, $\operatorname{supp} S \subset A$ and $S(\varphi)$ is $\mathcal{C}^\infty$ for $\mathcal{C}^\infty$ form φ on U (if it is well-defined). Therefore if $u := S(\omega)$ is well-defined on $\mathbb{C}^n$ then $\overline{\partial} u = \omega$.

We have

$$\int_U S(\zeta, z) \wedge \omega(\zeta) = \int_{A_z \cap \overline{\Omega} \cap U'} S(\zeta, z) \wedge \omega(\zeta).$$

If $z \in \mathbb{C}^n$ is such that there is $w \in A_z \cap \overline{\Omega} \cap U'$ then $F(z) = F(w)$, $A_z = A_w$, so $A_z \cap \overline{\Omega} = A_w \cap \overline{\Omega}$ is compact. It follows that $A_z \cap \overline{\Omega} \cap U' \subset\subset U$, hence the integral is well-defined.

In the opposite case (e.g. if $\operatorname{Re} F(z) > 0$) u vanishes. Recall that $\overline{\partial} u = \omega = 0$ on a domain $\mathbb{C}^n \setminus \overline{\Omega}$ (as $\partial\overline{\Omega} = \partial\Omega$ is connected) containing $\{\operatorname{Re} F > 0\}$, so $u = 0$ on $\mathbb{C}^n \setminus \overline{\Omega}$ (and on $\mathbb{C}^n \setminus \Omega$). □

References

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